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WAVES IN THE ATMOSPHERE AND OCEANS.

By

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WAVES IN THE ATMOSPHERE AND OCEANS

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LECTURE ONE SOME WAVE CONCEPTS

1. Water waves

The aim of this lecture is to introduce some of the concepts related to the theory of waves in fluids. Surface waves on water will be used to illustrate the definitions.

The Navier-Stokes equations for a non-rotating fluid may be written in the form

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} = -\frac{1}{\rho} \underset{\sim}{\nabla} p + \underset{\sim}{g} \qquad (1)$$

Together with the continuity equation

$$\frac{D\rho}{Dt} + \rho \, \underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0 \qquad (2)$$

these describe the motion of the fluid when the boundary conditions are specified. (The symbols are defined in the table of symbols).

Assume that the water is incompressible and uniform, so that

$$\frac{D\rho}{Dt} = 0 .$$

Equation (1) contains non-linear terms. The theory is simplified by assuming that the motion consists of small perturbations about a basic state (in this case a state of rest) and then neglecting products of small quantities. This method of approximation is known as linearising the equations. When this has been done the equations become

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad ; \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad ; \quad \frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (3)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 . \qquad (4)$$

Eliminating p from equations (3) leads to

$$\frac{\partial \underset{\sim}{\chi}}{\partial t} = \underset{\sim}{0}$$

where

$$\underset{\sim}{\chi} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} , \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} , \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

is the vorticity of the flow. Thus $\underset{\sim}{\chi}$ remains unchanged by the fluid motions, and only the irrotational components of the velocity need be considered in what follows. The velocity may thus be represented by a potential function ϕ such that

$$\underset{\sim}{u} = \underset{\sim}{\nabla} \phi \tag{5}$$

subject to the constraint imposed by the continuity equation

$$\nabla^2 \phi = 0 \tag{6}$$

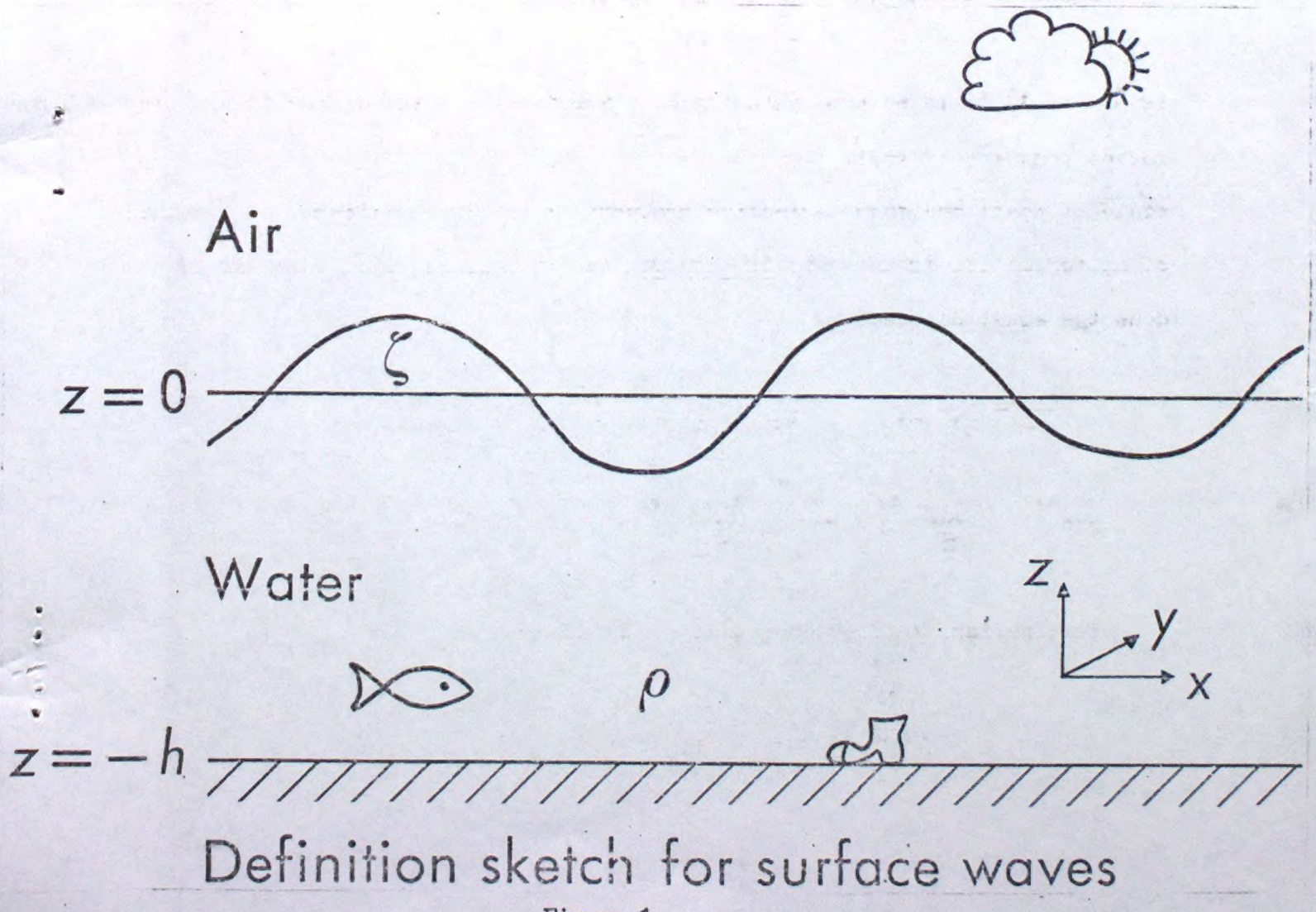
This is Laplace's equation, and its solution is governed by the boundary conditions.

The problem for water waves is illustrated in figure 1. At the bottom the boundary condition is

$$w = \frac{\partial \phi}{\partial z} = 0 \qquad \text{at} \quad z = -h \tag{7}$$

and that at the upper surface is

$$w = \frac{D}{Dt} \zeta = \frac{\partial \phi}{\partial z} \qquad \text{at} \quad z = \zeta \tag{8}$$



Definition sketch for surface waves

Figure 1

Equation (3) also imposes a condition on ϕ, namely that

$$\nabla \frac{\partial \phi}{\partial t} = -\frac{1}{\rho} \nabla p$$

so that, absorbing the constant of integration into ϕ,

$$\frac{\partial \phi}{\partial t} = -\frac{1}{\rho} p$$

and, since the pressure perturbation at $z=0$ is $\rho g \zeta$,

$$\frac{\partial \phi}{\partial t} = -g\zeta \qquad \text{on} \quad z = 0 \qquad (9)$$

If both ζ and $w = \frac{\partial \phi}{\partial z}$ are assumed to be small then, to the same accuracy as (3) and (4), the boundary condition (8) may be linearised and this, together with (7) and (9) may be applied on $z=0$. Combining (8) with (9) then gives

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \qquad \text{on} \quad z = 0. \qquad (10)$$

Now seek sinusoidal solutions of the form

$$\phi = \mathrm{Re}\{ A(z) \exp i(\underset{\sim}{k}.\underset{\sim}{x} - \omega t)\} \qquad (11)$$

where A is the amplitude, ω the frequency, and $\underset{\sim}{k} = (k_x, k_y, 0)$ the wave vector with wave numbers k_x and k_y in the x and y directions respectively. The wavelength is $2\pi/|\underset{\sim}{k}|$ and the phase speed is $c = \omega/|\underset{\sim}{k}|$. Any wave of the form $b \exp i(\underset{\sim}{k}.\underset{\sim}{x} - \omega t)$, for general $\underset{\sim}{k}$, is called a plane wave. An arbitrary surface water wave pattern may, if the assumptions in the linearisation process above are satisfied, be constructed by Fourier synthesis using plane waves of different wave vectors.

(11) must satisfy (6), so that

$$\frac{d^2 A}{dz^2} - k^2 A = 0$$

and hence $$A(z) = A_c \cosh k(z+h) + A_s \sinh k(z+h) \qquad (12)$$

where $k = |\underset{\sim}{k}|$ and in order to satisfy (7) $A_s = 0$. The solution thus far is

$$\phi = \mathrm{Re}\{ A_c \cosh k(z+h) \exp i(\underset{\sim}{k}.\underset{\sim}{x} - \omega t)\}.$$

(it will often be assumed in the following lectures that the real part of a complex quantity is to be taken and, following convention, this will not always be stated explicitly).

ω is determined from (10), and is given by

$$\omega^2 = gk \tanh kh . \tag{13}$$

An expression such as (13) which determines the frequency as a function of the wave number is called a dispersion relation. Waves for which ω/k is not constant are said to be dispersive. Thus water waves are dispersive. Sound waves, for which the phase speed is independent of wavenumber, are said to be non-dispersive.

Two approximations to (13) are often made. The first, for shallow water, assumes that h is small compared with $2\pi/k$, and so

$$\omega \sim k (gh)^{1/2} \tag{14a}$$

The other, for deep water, assumes that h is large, so that $\tanh kh \sim 1$ and

$$\omega \sim (gk)^{1/2} \tag{14b}$$

The former is a good approximation for $h < 0.07\lambda$, where $\lambda = 2\pi/k$, when SI units are used and the latter for $h > 0.28\lambda$. (Lighthill, 1978).

Before proceeding the solution will be written in a more convenient form. The surface height is $\zeta = -\frac{1}{g}\frac{\partial \phi}{\partial t}$, so that

$$\zeta = \frac{i}{g}\omega A_c \cosh kh \exp i(\underline{k}.\underline{x} - \omega t)$$

Thus the amplitude of the surface wave is $a = \frac{\omega}{g}\cosh(kh) A_c$. The solution may thus be written as

$$\phi = \frac{ag}{\omega}\frac{\cosh k(z+h)}{\cosh kh} \exp i\{\underline{k}.\underline{x} - \omega t\} \tag{15}$$

where $\omega = (gk \tanh(kh))^{1/2}$.

Consider the mean momentum transport by the wave. This is

$$\langle \underset{\sim}{M} \rangle = \left\langle \int_{-h}^{\zeta} \rho \underset{\sim}{u} \, dz \right\rangle$$

where $$\langle \underset{\sim}{b} \rangle = \frac{k}{2\pi} \int_0^{\frac{2\pi}{k}\underset{\sim}{k}} \underset{\sim}{b} \cdot d\underset{\sim}{x} .$$

Thus

$$\langle \underset{\sim}{M} \rangle = \left\langle \int_{-h}^{0} \rho \underset{\sim}{u} \, dz \right\rangle + \left\langle \int_0^{\zeta} \rho \underset{\sim}{u} \, dz \right\rangle$$

$$= 0 + \left\langle \int_0^{\zeta} \rho \underset{\sim}{u} \, dz \right\rangle$$

$$\approx \rho \langle \zeta \underset{\sim}{u} \rangle \qquad \text{to second order accuracy.}$$

Substituting from (15) gives

$$\langle \underset{\sim}{M} \rangle = \frac{\rho g a^2}{2c} \underset{\sim}{k} \tag{16}$$

where $c|\underset{\sim}{k}| = \omega$.

There is thus a mean momentum transfer in the direction of propagation of the wave. (This is due to the 'Stokes drift'. See LeBlond and Mysak, 1978). The mean energy density per unit horizontal area is also a quadratic quantity and likewise may be expected to be non-zero. It is convenient to divide the energy into two components.

kinetic energy $$\left\langle \int_{-h}^{\zeta} \tfrac{1}{2} \rho \, \underset{\sim}{u} \cdot \underset{\sim}{u} \, dz \right\rangle = \tfrac{1}{4} \rho g a^2 \tag{17}$$

perturbation potential energy $$\left\langle \int_{-h}^{\zeta} \rho g z \, dz - \int_{-h}^{0} \rho g z \, dz \right\rangle = \tfrac{1}{4} \rho g a^2 \tag{18}$$

Note that the wave energy is partitioned equally between the kinetic energy and the potential energy. It is also interesting to note that the mean momentum transport is 1/c times the wave energy density, a phenomenon which is present in many waves (LeBlond and Mysak, 1978).

It is readily seen from (15) that the wave crests travel at the phase speed c. However, what is not so obvious is that the wave energy is not transported at this speed. Consider the energy density of the wave

$$E = \int_{-h}^{\zeta} \tfrac{1}{2} \rho \, \underset{\sim}{u} \cdot \underset{\sim}{u} \, dz + \int_0^{\zeta} \rho g z \, dz .$$

If this is propagated with velocity $\underset{\sim}{V}$, then

$$\frac{\partial E}{\partial t} + \underset{\sim}{V} . \underset{\sim}{\nabla} E = 0 \qquad (19)$$

However

$$\underset{\sim}{u} = \frac{ag}{\omega} \frac{\cosh k(z+h)}{\cosh kh} \sin (\underset{\sim}{k} . \underset{\sim}{x} - \omega t) \, \underset{\sim}{k}$$

and $$\zeta = - a \cosh k(z+h) \sin (\underset{\sim}{k} . \underset{\sim}{x} - \omega t)$$

when the phase is chosen suitably.

(19) then becomes

$$(-\omega + \underset{\sim}{V} . \underset{\sim}{k}) \left\{ \int_{-h}^{\zeta} \frac{\rho}{4} \underset{\sim}{k} . \underset{\sim}{k} \frac{a^2 g^2 \cosh^2 k(z+h)}{\omega^2 \cosh^2 kh} \sin 2(\underset{\sim}{k} . \underset{\sim}{x} - \omega t) \, dz \right.$$
$$\left. + \tfrac{1}{4} \int_{0}^{\zeta} a^2 \cosh^2 k(z+h) \sin 2(\underset{\sim}{k} . \underset{\sim}{x} - \omega t) \, dz \right\} = 0$$

which is only an identity if

$$\omega = \underset{\sim}{V} . \underset{\sim}{k}$$

ie if $$\frac{d\omega}{d\underset{\sim}{k}} = \left(\frac{\partial \omega}{\partial k_1} , \frac{\partial \omega}{\partial k_2} , \frac{\partial \omega}{\partial k_3} \right) = \underset{\sim}{V} . \qquad (20)$$

$\underset{\sim}{V}$ is known as the <u>group velocity</u>, and for general plane waves of the form $b \exp i(\underset{\sim}{k} . \underset{\sim}{x} - \omega t)$ is given by (20).

For the deep water approximation, (15),

$$\frac{d\omega}{dk} = \frac{c}{2}$$

and the energy propagates at half the phase speed. For a description of a manifestation of this see Lighthill (1978. p 240). The dispersion relation for shallow water waves is given by (14), so that ω/k is a constant. These waves are thus non-dispersive, and so the group velocity is equal to the phase speed. Thus energy propagates at the phase speed.

1. <u>Exercise</u>

What is the correct solution if the water is inifinitely deep? (Hint: it must be finite as $z \to -\infty$).

What are the particle trajectories of the waves described above? (See Lighthill, 1978. He also discusses these waves when modified by surface tension).

2. Rossby waves

Rossby waves are barotropic waves which occur in both the atmosphere and oceans. In their pure form they are described by the barotropic vorticity equation which is derived below. Assume that the fluid motions are horizontal and that the fluid is horizontally homogeneous. The equations of motion are then

$$\rho\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv\right\} = -\frac{\partial p}{\partial x} \qquad (1)$$

$$\rho\left\{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu\right\} = -\frac{\partial p}{\partial y} \qquad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3)$$

where it is assumed that motion takes place on a <u>β - plane</u> with $f = f_0 + \beta y$ and $\beta y \ll f_0$.

Eliminating p from (1) and (2), and using (3) to write $u = -\frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$, gives

$$\frac{\partial}{\partial t}(\nabla^2\psi) + J(\psi, \nabla^2\psi) + \beta\frac{\partial \psi}{\partial x} = 0 \qquad (4)$$

This equation is non-linear. However, a wave of the form

$$\psi = a\cos(kx + ly - \omega t) \qquad (5)$$

is a solution of (4) if

$$\omega = -\beta k / (k^2 + l^2) \qquad (6)$$

Note that had (4) been linearised the result would have been the same. The addition of a constant wind to the problem by writing

$$\psi = -Uy + a\cos(kx + ly - \omega t) \tag{7}$$

leads to the dispersion relation

$$\omega = Uk - \beta k / (k^2 + l^2) \tag{8}$$

and in this case there is a stationary wave for which

$$U = \beta / (k^2 + l^2) \tag{9}$$

The group velocity of the waves is given by

$$\underset{\sim}{V} = \left(U - \frac{\beta(k^2 - l^2)}{(k^2+l^2)^2}, \frac{2\beta kl}{(k^2+l^2)^2}, 0\right)$$

and so the group velocity relative to the flow is

$$\underset{\sim}{V}' = \left(-\frac{\beta(k^2 - l^2)}{(k^2+l^2)^2}, \frac{2\beta kl}{(k^2+l^2)^2}, 0\right) \tag{10}$$

Consideration of the full spherical geometry of the Earth together with the full variation of the Coriolis parameter leads to Rossby-Haurwitz waves. The assumption on the sphere corresponding to a constant zonal wind on a β plane is that of solid body rotation. Under these conditions the solutions of the non-linear equations are again the same as those of the corresponding linearised equations (Neamtan. 1946). A review of the basic theory of Rossby waves on the sphere may be found in Longuet-Higgins (1964).

References

1. Water waves

LeBlond P H, Mysak L A — 1978 — Waves in the Ocean. Elsevier Scientific Publishing Co Amsterdam 602 pp.

Lighthill M J — 1978 — Waves in Fluids. Cambridge University Press Cambridge 504 pp.

2. Rossby waves

Longuet-Higgins M S — 1964 — Planetary waves on a rotating sphere. Proc. Roy. Soc. Series A 279 pp 446-473.

Neamtan, S M — 1946 — The motion of harmonic waves in the atmosphere. J. Met. 3 pp 53-56.

Symbols used in lecture 1

$\underset{\sim}{u} = (u, v, w)$	velocity
$\underset{\sim}{v} = (u, v, 0)$	horizontal velocity
$\underset{\sim}{V}$	group velocity
$\underset{\sim}{k} = (k_x, k_y, k_z)$ $\hat{k} = \lvert\underset{\sim}{k}\rvert$	wave vector
ω	frequency
c	phase speed
ρ	density
p	pressure
J	vertical displacement
$\underset{\sim}{g} = (0, 0, -g)$	gravitational acceleration
$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$	
$\underset{\sim}{\chi}$	vorticity
f	Coriolis parameter
$\beta = \frac{\partial f}{\partial y}$	variation of Coriolis parameter
f_0	reference value of Coriolis parameter
$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$	Jacobian of a and b
ψ	stream function
U	basic state velocity
k, l	wave numbers in x and y directions.

LECTURE TWO. THE EADY WAVE

1. Linearised Quasi-geostrophic Equations

Quasi-geostrophic theory has been discussed in a previous series of lectures. The governing equation for such motion is

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{v}_g \cdot \nabla\right] \zeta_g = f \frac{\partial w}{\partial z} - v \frac{\partial f}{\partial y} \tag{1}$$

and the thermodynamic equation takes the form

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{v}_g \cdot \nabla\right] \frac{g}{\theta_0} \theta = -N^2 w \tag{2}$$

where $N^2 = \frac{g}{\theta_0} \frac{d\Theta}{dz}$ is the Brunt-Väisälä frequency and Θ is a constant standard distribution of potential temperature.

Consider a geostrophic basic state on a f-plane determined by

$$\begin{aligned} \underset{\sim}{v} &= (\bar{U}(z), 0, 0) \\ \theta &= \Theta(y, z) \end{aligned} \tag{3}$$

$\bar{U}$ and Θ must satisfy the thermal wind relation

$$f \frac{d\bar{U}}{dz} = -\frac{g}{\theta_0} \frac{\partial \Theta}{\partial y} \tag{4}$$

Subsequent algebra is simplified if it is now assumed that $\frac{d\bar{U}}{dz}$ is constant as is N^2.

Linearising (1) and (2) about this basic state gives

$$\left[\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right] \zeta_g' = f \frac{\partial w'}{\partial z}$$

and

$$\left[\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right] \frac{g}{\theta_0} \theta' = -N^2 w + f v' \frac{\partial \bar{U}}{\partial z} \tag{5}$$

where a prime denotes a perturbation quantity. Eliminating w' from (5) gives

$$\left[\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right] f\zeta_g' = -f^2 \frac{\partial}{\partial z}\left[\frac{1}{N^2}\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)\frac{g\theta'}{\theta_0} - v' f \frac{\partial \bar{U}}{\partial z}\right] \tag{6}$$

The geostrophic velocity corresponding to a geopotential distribution is

$$u' = -\frac{1}{f}\frac{\partial \phi}{\partial y}, \quad v' = \frac{1}{f}\frac{\partial \phi}{\partial x} \tag{7}$$

and the corresponding geostrophic vorticity is $\zeta_g = \frac{1}{f}\nabla^2\phi$. (8)
The hydrostatic equation allows θ to be determined as a function of ϕ, namely

$$\theta = \frac{\theta_0}{g}\frac{\partial \phi}{\partial z} \tag{9}$$

Substituting into (6) from (7) - (9) gives

$$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right) q' = 0 \qquad (10)$$

where

$$q' = \phi_{xx} + \phi_{yy} + \frac{f^2}{N^2}\phi_{zz} \qquad (11)$$

is the perturbation potential vorticity for this problem.

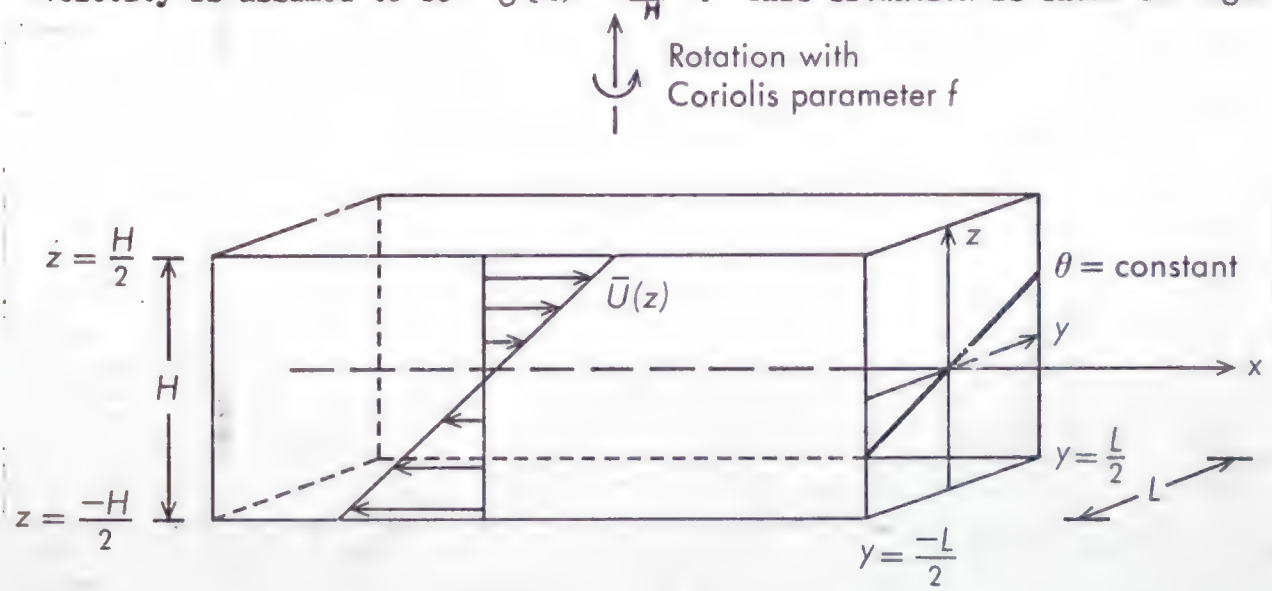
The time derivatives may be eliminated from (5) to give

$$N^2 \nabla^2 w' + f^2 \frac{\partial^2 w'}{\partial z^2} = 2\frac{\partial \bar{U}}{\partial z}\frac{\partial}{\partial x}(\nabla^2 \phi') \qquad (12)$$

This is the ω -equation for the present system (Hoskins et al, 1978), and may be used to determine the vertical velocity if ϕ' is known.

2. Eady Baroclinic Instability

Consider motion of a fluid governed by (10) and (11) of the previous section which is confined to a channel of width L and height H . The basic state velocity is assumed to be $\bar{U}(z) = \frac{Uz}{H}$. This situation is shown in figure 1.



The channel for the Eady model.

Figure 1.

The chosen profile for the wind corresponds to a uniform gradient of potential temperature across the channel. Eady (1949) considered the solutions of (10) above in the special case of initially uniform zero q' . At all subsequent times q' is zero as a direct result of the governing equation.

Non-dimensional coordinates

$$\frac{x}{L_R} \;,\; \frac{y}{L_R} \;,\; \frac{z}{H} \;,\; \frac{t}{(NH/fU)} \qquad \text{with } L_R = \frac{NH}{f} \tag{1}$$

transform (10) above into

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{2}$$

where x, y and z now represent non-dimensional coordinates.

The boundary condition at the top and bottom of the channel is $w=0$ and this substituted into the thermodynamic equation gives

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\phi'_z - \phi'_x = 0 \qquad \text{on } z = \pm\tfrac{1}{2}\,. \tag{3}$$

On the lateral boundaries $v' = 0$, so that

$$\frac{\partial \phi'}{\partial x} = 0 \qquad \text{on } y = \pm\tfrac{1}{2}\,\frac{L}{L_R}\,. \tag{4}$$

It should be noted that the governing equation (2) is again Laplace's equation (see Lecture 1, section 1) and the time dependence of the solutions is a result of the boundary conditions.

Seek sinusoidal solutions of the form

$$\phi' = A\gamma(z)\exp i(kx - \omega t)\sin(m\pi y) \tag{5}$$

where $m = \frac{L_R}{L}\,n$, and n is an integer

Then (2) requires that

$$\frac{d^2\gamma}{dz^2} - (k^2 + m^2\pi^2)\gamma = 0$$

and hence

$$\gamma(z) = a\sinh(k_m z) + b\cosh(k_m z) \tag{6}$$

where $$k_m = (k^2 + m^2\pi^2)^{1/2} \tag{7}$$

and a and b are arbitrary complex numbers which will be determined from the boundary conditions.

Imposing the boundary conditions on $z = \pm\frac{1}{2}$ gives

$$\omega\left(\frac{a}{b}\right) = \left(\frac{k}{2}\right)\tanh\left(\frac{k_m}{2}\right) - \frac{k}{k_m} \tag{8}$$

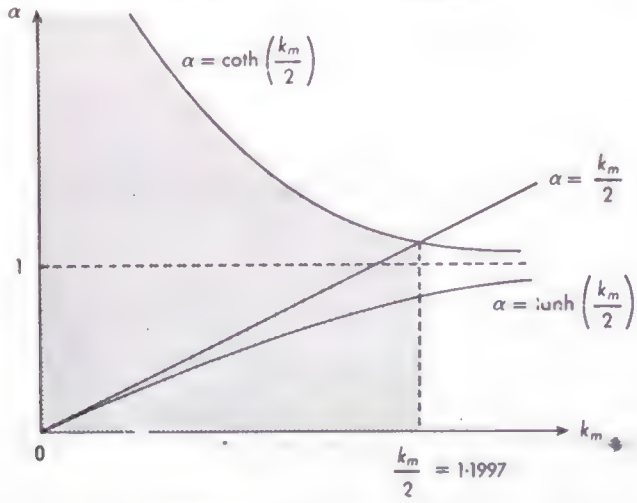
$$\omega\left(\frac{b}{a}\right) = \left(\frac{k}{2}\right)\coth\left(\frac{k_m}{2}\right) - \frac{k}{k_m} \tag{9}$$

which yield

$$\omega^2 = -\frac{k}{k_m}\left[\frac{k_m}{2} - \tanh\left(\frac{k_m}{2}\right)\right]\left[\coth\left(\frac{k_m}{2}\right) - \frac{k_m}{2}\right] \qquad (10)$$

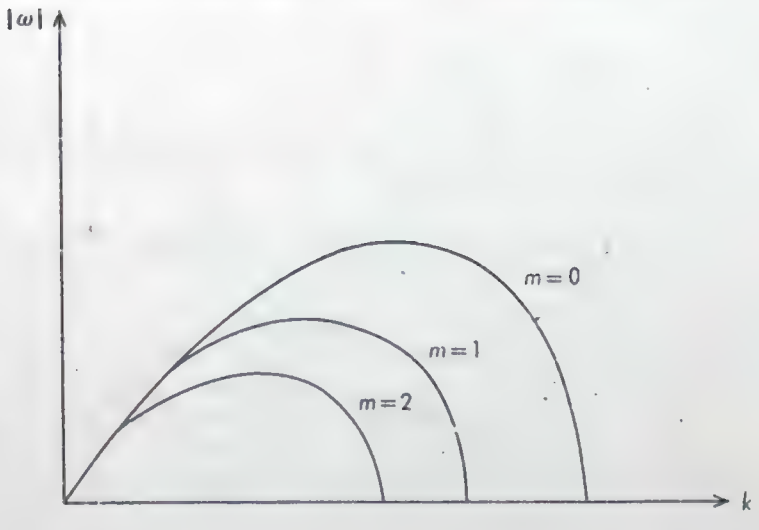
If $\omega^2<0$ the wave is growing or decaying and is stationary relative to the chosen coordinate system. If $\omega^2>0$ the wave travels with phase speed ω/k. In order that the wave may grow $-\omega^2>0$. The shaded area in figure 2 shows the region in which this relation is valid. Thus waves with total wavenumber less than 2.3994 can grow or decay, whereas those of greater wavenumber (shorter wavelength) are stable. The variation of the growth rate with zonal wave number and m is shown in figure 3. Substituting values appropriate to the atmosphere into the equations gives a 'short wave cut-off' at about 2500 km, with the growth rate maximum at $k_m = 1.6062$, which corresponds to 4000 km. The associated e-folding time is 1.1 days.

Instead of solving (8) and (9) for ω they may be solved for a/b. This gives the equation



Determination of the sign of ω^2.
$\omega^2<0$ in the shaded area.

Figure 2.



Schematic growth-rate curves for Eady baroclinic waves.

Figure 3.

$$\left(\frac{a}{b}\right)^2 = -\tanh^2\left(\frac{k_m}{2}\right)\left[\frac{\coth\left(\frac{k_m}{2}\right) - \left(\frac{k_m}{2}\right)}{\frac{k_m}{2} - \tanh\left(\frac{k_m}{2}\right)}\right]$$

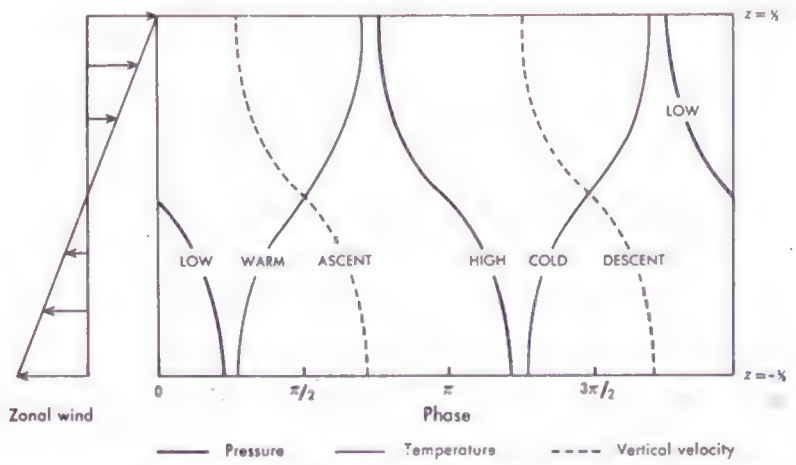
For a growing wave $Im(\omega) > 0$ so that, in order to satisfy (8), $Im(a/b) > 0$.

Thus

$$\frac{a}{b} = i\tanh\left(\frac{k_m}{2}\right)\left[\frac{\coth\left(\frac{k_m}{2}\right) - \frac{k_m}{2}}{\frac{k_m}{2} - \tanh\left(\frac{k_m}{2}\right)}\right]^{\frac{1}{2}} = iK, \text{ say,} \qquad (10)$$

and the corresponding solution, after a suitable choice of origin, is

$$\phi' = \exp(-i\omega t)\, b\left[\cosh(k_m z)\cos(kx) - K\sinh(k_m z)\sin(kx)\right]\sin(m\pi y) \qquad (11)$$



Sketch of the phases of the variables for the most unstable Eady model.

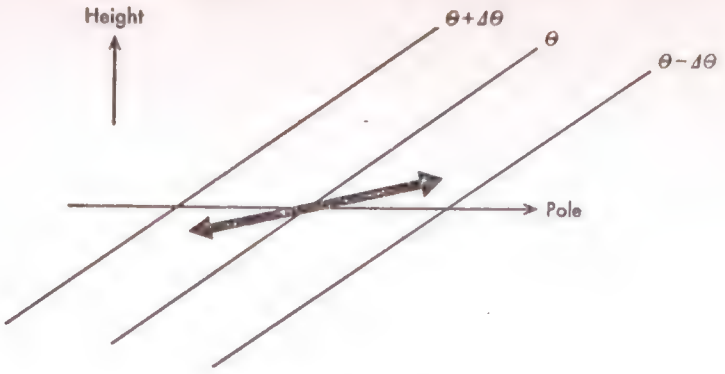
Figure 4.

The phases of the physical quantities for the most unstable mode are sketched in figure 4. The vertical velocity may be derived from (12) of the previous section. The relative phases of the meteorological variables were first discussed by Eady (1949). As in growing disturbances in the atmosphere the presssure wave slopes westwards with height. It can also be seen that the potential temperature wave slopes eastwards with height. This latter is not often commented upon in texts, which usually imply that all meteorological variables slope westwards with height in the growing wave. The vertical velocity wave slopes westward with height and leads the temperature wave in the lower atmosphere.

Baroclinic waves grow by converting the available potential energy of the basic state to kinetic energy of the growing eddy. In the course of this they transport heat polewards, thus acting to destroy the temperature structure responsible for their growth. A quantitative measure of the northward heat transport at a given level is

$$H = \frac{\int_0^{\frac{2\pi}{k}} \theta' v' \, dx}{(2\pi/k)} = \frac{b^2 e^{-i\omega t}}{2f\theta_0}\, g k k_m K \sin^2(m\pi y)$$

which is independent of height. Note that the sign is such as to move warm air northwards and cold air southwards. This transport of heat means that the air parcels cannot travel along the isentropes. Instead, they follow the trajectories sketched in figure 5. It is possible to derive the slope of the



Particle trajectories for growing baroclinic wave.

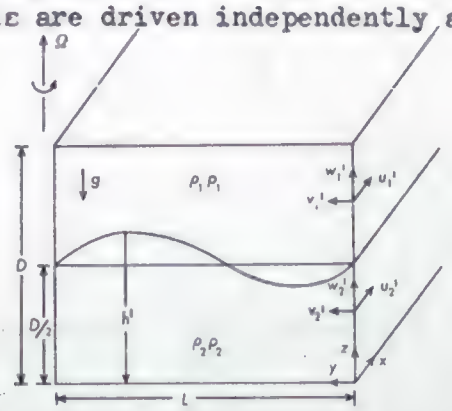
Figure 5.

trajectories from the Eady solution, although to do so requires that the ω-equation be solved. The result of such a calculation shows that the trajectories of parcels for the most rapidly growing mode bisect the angle between the horizontal and the isentropes. Knowledge of the vertical velocity enables the vertical heat flux to be determined. This is found to be such as to increase the static stability of the basic state; the waves transport warm air upwards and cold air descends.

It is interesting to notethat when the Eady mode is independent of y it also constitutes a solution of the non-linear equations (10) and (11) of the previous section for the present case of constant f . This is a physically reasonable solution since for an unbounded domain the heat transport by the wave can never alter the symmetry of the problem and the basic state is effectively left unchanged by the wave. When the problem is of finite north-south extent the Eady solution is no longer a solution of the non-linear equations, reflecting the tendency of the Eady to alter the basic state and to draw its energy from the finite supply of the initial conditions.

3. The Two Layer Model

This section gives a brief description of the two-layer model of baroclinic instability used by Pedlosky (1970). The problem is summarised in figure 6. The upper and lower levels are driven independently at different rates by rotating



The two layer model. (After Pedlosky, 1970).

Figure 6.

the lid of the apparatus. This problem has been modelled in the laboratory in the present form, whereas the Eady problem is an approximation to laboratory experiments.

Following Pedlosky (1970), non-dimensional variables are used. The scalings are defined in the Appendix. The equations of motion reduce to

$$\varepsilon\left[\frac{\partial u_n}{\partial t} + u_n\frac{\partial u_n}{\partial x} + v_n\frac{\partial u_n}{\partial y} + w_n\frac{\partial u_n}{\partial z}\right] - (1+\varepsilon\beta y)v_n = -\frac{\partial p_n}{\partial x} + \frac{E}{2}\nabla_S^2 u_n \qquad (1)$$

$$\varepsilon\left[\frac{\partial v_n}{\partial t} + u_n\frac{\partial v_n}{\partial x} + v_n\frac{\partial v_n}{\partial y} + w_n\frac{\partial v_n}{\partial z}\right] + (1+\varepsilon\beta y)u_n = -\frac{\partial p_n}{\partial y} + \frac{E}{2}\nabla_S^2 v_n \qquad (2)$$

$$\delta^2\varepsilon\left[\frac{\partial w_n}{\partial t} + u_n\frac{\partial w_n}{\partial x} + v_n\frac{\partial w_n}{\partial y} + w_n\frac{\partial w_n}{\partial z}\right] = -\frac{\partial p_n}{\partial z} + \frac{\delta^2 E}{2}\nabla_\delta^2 w_n \qquad (3)$$

$$\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y} + \frac{\partial w_n}{\partial z} = 0 \qquad (4)$$

for $n = 1, 2$

subject to the boundary conditions

$$\frac{\varepsilon}{2}F\left[\frac{\partial h}{\partial t} + u_n\frac{\partial h}{\partial x} + v_n\frac{\partial h}{\partial y}\right] = w_n \qquad \text{at the interface} \qquad (5)$$

$$v_n = 0 \qquad \text{on } y = 0,\ 1 \qquad (6)$$

and $$\lim_{X\to\infty}\frac{1}{2X}\int_{-X}^{X}\frac{\partial u_n}{\partial t}\,dx = 0 \qquad \text{on } y = 0,\ 1. \qquad (7)$$

The last boundary condition is introduced by an approximation which will be made below, but is included here for convenience (Phillips, 1954).

Writing the variables in the form

$$\begin{aligned}
u_n &\sim u_n^{(0)} + \varepsilon u_n^{(1)} + \varepsilon^2 u_n^{(2)} + \cdots \\
v_n &\sim v_n^{(0)} + \varepsilon v_n^{(1)} + \varepsilon^2 v_n^{(2)} + \cdots \\
w_n &\sim \varepsilon w_n^{(1)} + \varepsilon^2 w_n^{(2)} + \cdots \\
h &\sim h^{(0)} + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)} + \cdots \\
p_n &\sim p_n^{(0)} + \varepsilon p_n^{(1)} + \varepsilon^2 p_n^{(2)} + \cdots
\end{aligned}$$

and substituting these into the equations gives O(1) terms

$$\begin{aligned}
u_n^{(0)} &= -\frac{\partial p_n^{(0)}}{\partial y} \\
v_n^{(0)} &= -\frac{\partial p_n^{(0)}}{\partial x} \\
0 &= \frac{\partial p_n^{(0)}}{\partial z} \\
\frac{\partial u_n^{(0)}}{\partial x} &+ \frac{\partial v_n^{(0)}}{\partial y} = 0 .
\end{aligned}$$

This is simply the geostrophic conditions. Collecting terms of $O(\varepsilon)$ gives

$$\left[\frac{\partial}{\partial t} + \frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial x}\right]\left[\nabla^2\psi_1 + F(\psi_2-\psi_1) + \beta y\right] = -r\nabla^2\psi_1 \qquad (8)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial\psi_2}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\psi_2}{\partial y}\frac{\partial}{\partial x}\right]\left[\nabla^2\psi_2 + F(\psi_1-\psi_2) + \beta y\right] = -r\nabla^2\psi_2 \qquad (9)$$

where $\psi_n = p_n^{(0)}$ and $r = k''/\varepsilon$. Note that Ekman pumping has been used (Pedlosky, 1970). These are the appropriate quasi-geostrophic equations for this problem.

Linearising (8) and (9) about a basic state

$$\psi_1 = -U_1 y \tag{10}$$

$$\psi_2 = -U_2 y \tag{11}$$

and seeking wave solutions of the form

$$\psi_1' = A \exp i(kx - ct) \sin(m\pi y) \tag{12}$$

$$\psi_2' = \gamma A \exp i(kx - ct) \sin(m\pi y) \tag{13}$$

gives

$$c = \frac{U_1 + U_2}{2} - \frac{(k_m^2 + F)}{(k_m^2 + 2F)}\left(\frac{\beta}{k_m^2} + \frac{i\Gamma}{k}\right)$$

$$\pm \frac{\left[(U_1 - U_2)^2 k_m^4 (k_m^4 - 4F^2) + 4F^2\left(\beta + i\Gamma \frac{k_m^2}{k}\right)^2\right]^{1/2}}{2k_m^2 (k_m^2 - 2F)} \tag{14}$$

and

$$\gamma = \frac{k_m^2 + F}{F} + \frac{\beta + F(U_1 - U_2)}{F(c - U_1)} + i \frac{k_m^2 \Gamma/k}{F(c - U_1)} . \tag{15}$$

Note that if $\Gamma = \beta = 0$ the analogue of the Eady mode is obtained and

$$c = \frac{U_1 + U_2}{2} \pm \frac{(U_1 - U_2)(k_m^4 - 4F^2)^{1/2}}{2(k_m^2 - 2F)}$$

Thus the growing waves travel with the mean speed of the flow and have a short wave cut off at $k_m^2 = 2F$.

The results of this last section will be used in the next lecture when finite amplitude baroclinic waves are discussed. The two-layer model has been discussed more widely in the literature of finite amplitude waves than has the Eady model.

APPENDIX

The scalings used in Pedlosky (1970) are as below, where a prime denotes a dimensional variable.

$(x, y) = (x', y')/L$ — horizontal coordinates

$z = z'/D$ — vertical coordinate

$(u_n, v_n) = (u'_n, v'_n)/U$ — horizontal components of velocity

$w_n = \left(\frac{L}{D}\right) w'_n/U$ — vertical component of velocity

$t = \left(\frac{U}{L}\right) t'$ — time

$P_1 = (p'_1 + \rho_1 g(z' - D))/(\rho_1 U f_0 L)$ } hydrostatic pressure

$P_2 = (p'_2 + \rho_2 g(z' - D/2) - \rho_1 g \frac{D}{2})/(\rho_2 U f_0 L)$ } hydrostatic pressure

$h = \left(h' - \frac{D}{2}\right) \Big/ \left(\rho_1 U f_0 L / g(\rho_2 - \rho_1)\right)$ — interface height

Motion is considered to be on a β -plane, so that

$$f = f_0 + \beta' y'$$

The following dimensionless parameters are used.

$\varepsilon = U/f_0 L$ — Rossby number

$E = 2\nu/f_0 D^2$ — Ekman number

$\beta = \beta' L^2/U$ — planetary vorticity factor

$\frac{\Delta\rho}{\rho} = \frac{(\rho_2 - \rho_1)}{\rho_1}$ — density ratio

$F = f_0^2 L^2 / \left(\frac{\Delta\rho}{\rho} g \frac{D}{2}\right)$ — internal Froude number

$\delta = D/L$ — aspect ratio

The operator ∇_δ^2 is defined by

$$\nabla_\delta^2 = \frac{\partial^2}{\partial z^2} + \delta^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

ν denotes the kinematic viscosity of the fluid.

Bibliography for Lecture 2

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3. Pedlosky J — 1970 — Finite amplitude baroclinic waves. J. Atmos. Sci. 27, 15-30.

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Table of Symbols for Lecture 2

$\underset{\sim}{v}_g = (u_g, v_g)$	geostrophic velocity
ζ_g	geostrophic vorticity
w	vertical velocity
θ	potential temperature
θ_o	representative value of θ
Θ	reference distribution of θ
g	gravitational acceleration
N	Brunt-Väisälä frequency
$\overline{U}$	basic state velocity
f	Coriolis parameter
q'	perturbation potential vorticity
$L_R = \frac{NH}{f}$	Rossby radius of deformation
$k_m = (k^2 + m^2\pi^2)^{\frac{1}{2}}$	total wave number
k	zonal wave number
$m\pi \frac{L_R}{L}$	meridional wave number
H	northward heat flux
K	amplitude ratio between cos and sin terms of a growing wave
ω	angular frequency of wave.

The symbols used in section 3 are defined in the Appendix.

LECTURE THREE WEAKLY NON-LINEAR BAROCLINIC WAVES

The previous lectures discussed linearised forms of the equations of motion. In the cases of Rossby waves and Eady waves on an infinite plane it was found that the solutions of the linearised equations were also 'linear' solutions of the non-linear equations due to the exact cancellation of the non-linear terms. Such behaviour is very uncommon, and the character of the solutions of a non-linear equation usually differs from that of the solutions of the corresponding linearised equation. The more usual situation is illustrated in the following example.

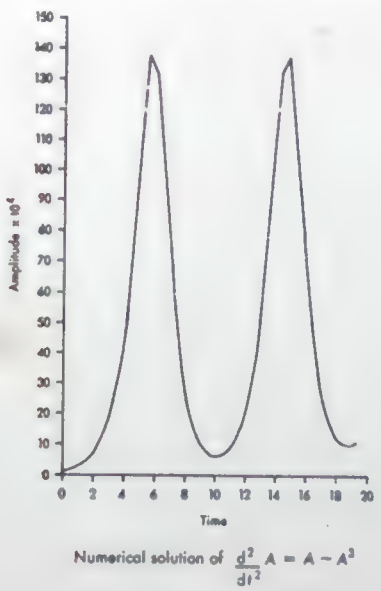
Consider the ordinary differential equation

$$\frac{d^2}{dt^2} A = a_1 A + a_2 A |A|^2 \qquad (1)$$

where a_1 and a_2 are constants, and A may be complex. If A is assumed to be infinitesimally small, then (1) may be linearised to give

$$\frac{d^2}{dt^2} A = a_1 A$$

which has solutions $A = b_1 \exp a_1^{\frac{1}{2}} t + b_2 \exp -a_1^{\frac{1}{2}} t$ for arbitrary b_1, b_2. If $a_1 > 0$ these solutions are exponential, the first term growing while the second term decays. Suppose that $b_2 = 0$. Then in the linearised solution A grows with time without limit. As this solution grows, the $a_2 A|A|^2$ term in (1) increases in magnitude and begins to move the solution away from that of the linearised equation. If $a_2 > 0$ the effect of this term is to increase the rate of growth beyond that of linear theory. If $a_2 < 0$ the 'linear' growth rate is reduced and may even by reversed. A solution of (1) with $a_1 = 1.0$, $a_2 = -1.0$ and an initial value for A of 0.01, which was calculated on COSMOS, is shown in figure 1. It can be seen that the effect of the non-linear term is to alter the exponential growth of the linearised equation solution to one in which A vacillates.



Numerical solution of $\frac{d^2}{dt^2} A = A - A^3$

Figure 1.

Equation (1) has not yet been linked with the problem of baroclinic instability.

It is the aim of this lecture to show how such an equation may be derived from the problems discussed in the previous lecture. The two-level model discussed by Pedlosky (1970) will be examined here. This has been studied more widely than the Eady model when the wave is assumed to be of finite amplitude, although the latter model has been investigated by Drazin (1970, 1972).

Consider the inviscid ($r=0$) forms of Pedlosky's equations (equations (8) and (9) of section 3 of the last lecture). Instability is possible if

$$U_1 - U_2 > U_c = \frac{2\beta F}{k_m^2 (4F^2 - k_m^4)^{\frac{1}{2}}} \qquad (2)$$

(see previous lecture). Consider a value of the shear slightly removed from this critical value such that

$$U_1 - U_2 - U_c = \Delta \qquad \text{where} \quad |\Delta| \ll |U_c| \qquad (3)$$

The growth rate is determined by the imaginary part of the phase speed, and, from (14) of section 3 of the previous lecture, is given by

$$\alpha c_i = \pm \left(\frac{\sqrt{3}\,\beta F}{k_m^2 (k_m^2 + 2F)} \right) \left(\frac{\Delta}{U_c} \right)^{\frac{1}{2}}.$$

Accordingly, the wave grows exponentially with growth rate proportional to $\Delta^{\frac{1}{2}} / U_c^{\frac{1}{2}}$. If this is much smaller than unity the time variation of the amplitude of the wave occurs on a much longer timescale than the sinusoidal variation of the wave. This results in the amplitude modulation being effectively decoupled from the wave oscillation. This may be represented by introducing a 'slow time'

$$T = |\Delta|^{\frac{1}{2}} t \qquad (4)$$

so that the variation of the wave amplitude is resolved by this, and the oscillations associated with the wave are dependent on t.

In order to avoid a proliferation of superscripts, write ϕ_n for ψ_n' of the previous lecture. When the slow time is considered

$$\phi_n = \phi_n(x, y, t, T) \qquad (5)$$

and the total stream function is

$$\psi_n = -U_n y + \phi_n(x, y, t, T). \qquad (6)$$

The equations of the previous lecture thus reduce to

$$\left[\frac{\partial}{\partial t} + |\Delta|^{\frac{1}{2}}\frac{\partial}{\partial \tau} + (U_2 + U_c + \Delta)\frac{\partial}{\partial x}\right]\left[\nabla^2\phi_1 + F(\phi_2 - \phi_1)\right]$$

$$+ \frac{\partial \phi_1}{\partial x}(\beta + FU_c + F\Delta) + J(\phi_1, \nabla^2\phi_1 + F(\phi_2 - \phi_1)) = 0$$

$$\left[\frac{\partial}{\partial t} + |\Delta|^{\frac{1}{2}}\frac{\partial}{\partial \tau} + U_2\frac{\partial}{\partial x}\right]\left[\nabla^2\phi_2 + F(\phi_1 - \phi_2)\right] \tag{6}$$

$$+ \frac{\partial \phi_2}{\partial x}(\beta - FU_c - F\Delta) + J(\phi_2, \nabla^2\phi_2 + F(\phi_1 - \phi_2)) = 0$$

where the Jacobian $J(a,b)$ is defined by $J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}$

Expand ϕ_n as a power series in $|\Delta|^{1/2}$. Then

$$\phi_n = |\Delta|^{\frac{1}{2}}\phi_n^{(1)} + |\Delta|\phi_n^{(2)} + |\Delta|^{3/2}\phi_n^{(3)} + O(|\Delta|^2) \tag{7}$$

Substitution of (7) into (6) and collecting terms of the same order in $|\Delta|^{1/2}$ leads to a series of problems, each of $O(|\Delta|^{\frac{n}{2}})$. Consider $n=0$. The corresponding problem is that of determining the basic flow. This is specified, so now consider the $O(|\Delta|^{1/2})$ problem.

Equations (6) then become

$$\left[\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right]\left[\nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})\right] + \frac{\partial \phi_2^{(1)}}{\partial x}(\beta - FU_c) = 0$$

$$\left[\frac{\partial}{\partial t} + (U_2 + U_c)\frac{\partial}{\partial x}\right]\left[\nabla^2\phi_1^{(1)} + F(\phi_2^{(1)} - \phi_1^{(1)})\right] + \frac{\partial \phi_1^{(1)}}{\partial x}(\beta + FU_c) = 0 \tag{8}$$

and the boundary conditions are

$$\frac{\partial \phi_1^{(1)}}{\partial x} = \frac{\partial \phi_2^{(1)}}{\partial x} = 0 \qquad \text{at} \quad y = 0, 1. \tag{9}$$

This leads to a solution

$$\phi_1^{(1)} = \mathrm{Re}\left[A(\tau)\exp i\alpha(x - ct)\sin(m\pi y)\right]$$

$$\phi_2^{(1)} = \mathrm{Re}\left[A(\tau)\gamma\exp i\alpha(x - ct)\sin(m\pi y)\right] \tag{10}$$

where

$$c = U_2 + \frac{U_c}{2} - \frac{\beta(k^2m + F)}{k^2m(k^2m + 2F)} \tag{11}$$

and

$$\gamma = \frac{k^2m + F}{F} - \frac{\beta + U_cF}{F(U_2 + U_c - c)}. \tag{12}$$

There is thus no phase shift with height at this order of approximation. The wave is neutrally stable. Note that this is just the solution determined

in the last lecture for $U_1 - U_2 = U_c$.

Proceeding now to the $O(|\Delta|)$ problem the equations become

$$\left[\frac{\partial}{\partial t} + (U_2+U_c)\frac{\partial}{\partial x}\right]\left[\nabla^2\phi_1^{(2)} + F(\phi_2^{(2)} - \phi_1^{(2)})\right] + \frac{\partial \phi_1^{(2)}}{\partial x}(\beta + FU_c)$$

$$= -J\left(\phi_1^{(1)}, \nabla^2\phi_1^{(1)} + F(\phi_2^{(1)} - \phi_1^{(1)})\right)$$
$$- \frac{\partial}{\partial T}\left[\nabla^2\phi_1^{(1)} + F(\phi_2^{(1)} - \phi_1^{(1)})\right] \qquad (13)$$

$$\left[\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right]\left[\nabla^2\phi_2^{(2)} + F(\phi_1^{(2)} - \phi_2^{(2)})\right] + \frac{\partial \phi_2^{(2)}}{\partial x}(\beta - FU_c)$$

$$= -J\left(\phi_2^{(1)}, \nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})\right)$$
$$- \frac{\partial}{\partial T}\left(\nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})\right)$$

These equations only differ from those of the previous order in the forcing terms.

Since the horizontal structures of $\phi_1^{(1)}, \phi_2^{(1)}$ and their Laplacians are the same, the Jacobians vanish identically. Thus the only remaining forcing term is that due to the amplitude variation, which is represented by the $\partial/\partial T$. If Δ were zero, the choice of the initial state such that $U_1 - U_2 = U_c$ would imply that the wave amplitude would not vary with time. However, since non-zero leads to a growing solution the effect of the $\partial/\partial T$ term must be investigated. The T-dependence is multiplicative, so that

$$\frac{\partial}{\partial T}\left[\nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})\right]$$

$$= \left(\frac{1}{A}\frac{dA}{dT}\right)\left[\nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})\right]$$

This term is of the same form as that which is operated on by $\partial/\partial t$ in (13) and $\phi_2^{(1)}$ and $\phi_1^{(1)}$ are solutions of the homogeneous associated with (13). Thus the solution of (13) might be expected to have a variation such as

$$\phi_1^{(2)} = t\,\phi_1^{(1)}, \qquad \phi_2^{(2)} = t\,\phi_2^{(1)}.$$

For large t this would violate the implicit assumption that $\phi_n^{(m)}$ is bounded, for if $\phi_n^{(m)}$ may be arbitrarily large the series (7) will not converge. A solution such as that described here is said to be 'secular'.

The solution will be continued without being concerned with this problem as it is possible to avoid it.

Seek a solution

$$\phi_n^{(2)} = Re\left[A_n^{(2)}(T)\exp i\alpha(x-ct)\sin(m\pi y)\right]. \qquad (14)$$

This leads to

$$\gamma A_1^{(2)} - A_2^{(2)} = -\frac{1}{i\alpha F}\frac{dA}{dT}\left[\frac{\beta + FU_c}{(U_2+U_c-c)^2}\right]$$

$$-A_1^{(2)} + \frac{1}{\gamma}A_2^{(2)} = -\frac{\gamma}{i\alpha F}\frac{dA}{dT}\left[\frac{\beta - FU_c}{(U_2-c)^2}\right]. \qquad (15)$$

Note that the left hand sides of (15) are linearly dependent, so that eliminating these

$$\frac{dA}{dT}\left[\frac{\beta+FU_c}{(U_2+U_c-c)^2}+\gamma^2\frac{(\beta-FU_c)}{(U_2-c)^2}\right]=0 \qquad (16)$$

If the present method of solution is valid, the term in brackets must be zero, for otherwise

$$\frac{dA}{dT}=0$$

which contradicts the assumption that A varies with T. Substituting for the variables in the above expression confirms that the bracketed term is indeed zero. Thus the approach taken is justified, and it is possible to find a particular integral of (13) of the form (14). The secularity feared above does not occur at this order of approximation. By assuming that the solution at a given order of $|\Delta|^{1/2}$ explains as much of the perturbation as is possible for such a solution, the homogeneous solution of (13) may be assumed the be absorbed into the $O(|\Delta|^{1/2})$ solution.

Either $A_1^{(2)}$ or $A_2^{(2)}$ may be determined from the other using (15). Then

$$A_2^{(2)}=\gamma A_1^{(2)}+\frac{1}{i\alpha F}\frac{dA}{dT}\left[\frac{\beta+FU_c}{(U_2+U_c-c)^2}\right] \qquad (17)$$

However, the wave form (14) is of the same form as that of (10). Thus, the same reasoning that allowed us to neglect the homogeneous solution allows either $A_1^{(2)}$ or $A_2^{(2)}$ to be set to zero. It does not matter which, so arbitarily set $A_1^{(2)}=0$. $A_2^{(2)}$ could have beer chosen, and, if it had been, the sole difference would have been an implied change of phase.

This order of solution describes the phase difference between the upper and lower levels. If the substitution

$$\alpha c_i=\frac{1}{A}\frac{dA}{dT}$$

is made, the solution at this stage becomes identical with that of the previous lecture.

In addition to the homogeneous solution of the same form as (10) there is another which is independent of x and t This is of the form

$$\phi_n^{(2)}=\Phi_n^{(2)}(y,T) \qquad n=1,2 \qquad (18)$$

and is required to remove the secular terms at the next order of approximation. This represents the alteration to the basic flow by the wave. The mechanism modelled here is the destruction of the basic state gradients by the wave.

Summarising, the complete solution so far is

$$\psi_1 = -U_1 y + |\Delta|^{1/2} \mathrm{Re}\left[A \exp i\alpha(x-ct) \sin(m\pi y)\right] + |\Delta|\ \Phi_1^{(2)}(y,T) + O(|\Delta|^{3/2})$$

(19)

$$\psi_2 = -U_2 y + |\Delta|^{1/2} \mathrm{Re}\left[\gamma A \exp i\alpha(x-ct) \sin(m\pi y)\right] + |\Delta|\left[\Phi_2^{(2)}(y,T) + X_2^{(2)}(x,y,t,T)\right] + O(|\Delta|^{3/2})$$

where $X_2^{(2)}(x,y,t,T) = \mathrm{Re}\left[\frac{1}{i\alpha F}\frac{dA}{dT}\left[\frac{\beta+FU_c}{(U_2+U_c-c)^2}\right]\exp i\alpha(x-ct)\sin(m\pi y)\right]$

Proceeding, the final stage in the solution is reached. The equations are

$$\left[\frac{\partial}{\partial t} + (U_2+U_c)\frac{\partial}{\partial x}\right]\left[\nabla^2\phi_1^{(3)} + F(\phi_2^{(3)}-\phi_1^{(3)})\right] + \frac{\partial\phi_1^{(3)}}{\partial x}(\beta+FU_c)$$
$$= -\frac{\partial}{\partial T}\left[\nabla^2\phi_1^{(2)} + F(\phi_2^{(2)}-\phi_1^{(2)})\right] - \frac{\partial}{\partial x}\phi_1^{(1)} F\frac{\Delta}{|\Delta|}$$
$$- \frac{\partial}{\partial x}\left[\nabla^2\phi_1^{(1)} + F(\phi_2^{(1)}-\phi_1^{(1)})\right]\frac{\Delta}{|\Delta|}$$
$$- J(\phi_1^{(2)}, \nabla^2\phi_1^{(1)} + F(\phi_2^{(1)}-\phi_1^{(1)}))$$
$$- J(\phi_1^{(1)}, \nabla^2\phi_1^{(2)} + F(\phi_2^{(2)}-\phi_1^{(2)})) \qquad (20)$$

$$\left[\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right]\left[\nabla^2\phi_2^{(3)} + F(\phi_1^{(3)}-\phi_2^{(3)})\right] + \frac{\partial\phi_2^{(3)}}{\partial x}(\beta-FU_c)$$
$$= -\frac{\partial}{\partial T}\left[\nabla^2\phi_2^{(2)} + F(\phi_1^{(2)}-\phi_2^{(2)})\right] + \frac{\partial\phi_2^{(1)}}{\partial x}F\frac{\Delta}{|\Delta|}$$
$$- J(\phi_2^{(2)}, \nabla^2\phi_2^{(1)} + F(\phi_1^{(1)}-\phi_2^{(1)}))$$
$$- J(\phi_2^{(1)}, \nabla^2\phi_2^{(2)} + F(\phi_1^{(2)}-\phi_2^{(2)})).$$

After substituting the solutions obtained already these result in equations of the form

$$\left(\frac{\partial}{\partial t} + (U_2+U_c)\frac{\partial}{\partial x}\right)\left[\nabla^2\phi_1^{(3)} + F(\phi_2^{(3)}-\phi_1^{(3)})\right] + \frac{\partial\phi_1^{(3)}}{\partial x}(\beta+FU_c)$$
$$= \mathrm{Re}\ i\alpha A\sin(m\pi y)\exp i\alpha(x-ct)\left[\beta - \frac{\partial\Phi_1^{(2)}}{\partial y}\frac{(\beta+FU_c)}{(U_2+U_c-c)}\right.$$
$$\left. - \frac{\partial}{\partial y}\left(\frac{\partial^2\Phi_1^{(2)}}{\partial y^2} + F(\Phi_2^{(2)}-\Phi_1^{(2)})\right)\right]$$
$$+ \frac{(\beta+FU_c)}{4(U_2+U_c-c)^2}\left(\frac{d|A|^2}{dT}\right)\pi m\sin(2m\pi y)$$
$$- \frac{\partial}{\partial T}\left[\frac{\partial^2}{\partial y^2}\Phi_1^{(2)} + F(\Phi_2^{(2)}-\Phi_1^{(2)})\right] \qquad (21)$$

$$\left[\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right]\left[\nabla^2\Phi_2^{(2)}+F(\Phi_2^{(2)}-\Phi_1^{(2)})\right]+\frac{\partial\Phi_1}{\partial x}(\beta-FU_c)$$

$$=\mathrm{Re}\; i\alpha\gamma A\exp i\alpha(x-ct)\sin(m\pi y)\left[C-\frac{\partial\Phi^{(2)}}{\partial y}\frac{(\beta-FU_c)}{(U_2-c)}\right.$$

$$\left.-\frac{\partial}{\partial y}\left(\frac{\partial^2\Phi_2^{(2)}}{\partial y^2}+F(\Phi_1^{(2)}-\Phi_2^{(2)})\right)\right]$$

$$-\frac{\beta+FU_c}{4(U_2+U_c-c)^2}\frac{d|A|^2}{dT}m\pi\sin(2m\pi y)$$

$$-\frac{\partial}{\partial T}\left[\frac{\partial^2}{\partial y^2}\Phi_2^{(2)}+F(\Phi_1^{(2)}-\Phi_2^{(2)})\right] \qquad (22)$$

where B and C are constants. For the full form of these equations see Pedlosky (1970). The inhomogeneous terms of (21) are of two types. The first varies sinusoidally in x and t, but the other is independent of these variables. Thus the last two terms of each equation of (21) must sum to zero to prevent a secularity which has a linear dependence on x and t. This is the reason for introducing the correction to the basic state flow at $O(|A|)$ for in the absence of the $\Phi_n^{(2)}$ terms this singularity would not be removable. Thus

$$\frac{\partial}{\partial T}\left[\frac{\partial^2}{\partial y^2}\Phi_1^{(2)}+F(\Phi_2^{(2)}-\Phi_1^{(2)})\right]=\frac{(\beta+FU_c)m\pi}{4(U_2+U_c-c)^2}\frac{d}{dT}|A|^2\sin(2m\pi y)$$

$$\frac{\partial}{\partial T}\left[\frac{\partial^2}{\partial y^2}\Phi_2^{(2)}+F(\Phi_1^{(2)}-\Phi_2^{(2)})\right]=-\frac{(\beta+FU_c)m\pi}{4(U_2+U_c-c)^2}\frac{d|A|^2}{dT}\sin(2m\pi y) \qquad (23)$$

Hence, imposing the boundary conditions at $y=0,1$ and also $\Phi(y,0)=0$, yields

$$\Phi_1^{(2)}=-\Phi_2^{(2)}=-\frac{\left[|A|^2-|A(0)|^2\right](\beta+FU_c)\,m\pi}{8(2m^2\pi^2+F)(U_2+U_c-c)^2}\left[\sin(2m\pi y)-\right.$$

$$\left.\frac{\sinh\left(\sqrt{2F}\,(y-1/2)\right)}{\cosh\sqrt{F/2}}\frac{m\pi}{\sqrt{F/2}}\right] \qquad (24)$$

and the correction to the zonal wind shear may be deduced from this to be

$$U_1^{(2)}-U_2^{(2)}=\frac{\partial}{\partial y}\left[\Phi_1^{(2)}-\Phi_2^{(2)}\right]$$

$$=\frac{\left[|A|^2-|A(0)|^2\right]m^2\pi^2(\beta+FU_c)}{2(2m^2\pi^2+F)(U_2+U_c-c)^2}\left[\cos(2m\pi y)-\right.$$

$$\left.\frac{\cosh\left(\sqrt{2F}\,(y-1/2)\right)}{\cosh\sqrt{F/2}}\right] \qquad (25)$$

This states that as the wave amplitude increases the shear of the zonal flow is reduced, thus leading to a negative feedback and the growth of the wave is slower than for the linear case. Should the wave amplitude fall below its initial value the zonal wind shear is increased. This leads to an

increased growth rate. The possibility of an oscillation of the wave amplitude thus arises.

Now that $\Phi_1^{(2)}$ and $\Phi_2^{(2)}$ are known as functions of A, the inhomogeneous terms in (21) may all be expressed in terms of A. Note that, as in the $O(|A|)$ equations, the terms involving B and C are of the same form as the homogeneous solution. Again, these could lead to a secularity unless they are identically zero. Pedlosky (1970) derives a condition for B and C to be zero, namely that

$$\frac{d^2 A}{d\tau^2} = \alpha^2 c_{oi}^2 A - \alpha^2 N A \left[|A(\tau)|^2 - |A(0)|^2 \right] \tag{26}$$

where

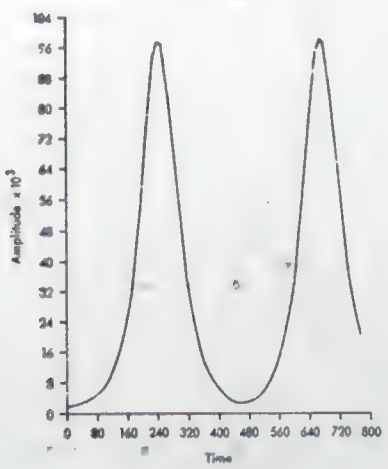
$$N = (2m^2\pi^2 + F)^{-1} \left[\frac{(\beta + F U_c)\, m^2 \pi^2 U_c}{8(k_m^2 + 2F)(U_2 + U_c - c)^2} \right] \times$$

$$\left[k_m^2 (2F - k_m^4) + 4m^2\pi^2 (k_m^2 - F) + (2F - k_m^4) \frac{4 \tanh \sqrt{F/2}}{\sqrt{F/2}} \left(\frac{2m^2\pi^2}{2m^2\pi^2 + F} \right) \right] \tag{27}$$

and

$$c_{oi}^2 = \frac{2\beta^2 F^2}{k_m^4 (k_m^2 + 2F)^2} \left(\frac{\Delta}{|\Delta| U_c} \right)$$

The properties of (26) have already been discussed, for it is of the same form as equation (1). Figure 2 shows the amplitude evolution for values of the parameters which are relevant to the atmosphere. Pedlosky (1970)



Amplitude variation of the most unstable wave for the ICAO atmosphere in a channel of width 10000 km and with a lid at 250 mb. [$U_1 = 43.5\ ms^{-1}$, $U_2 = 10\ ms^{-1}$, $U_c = 30.4\ ms^{-1}$, $c = 9.9\ ms^{-1}$, $\lambda = 8170$ km, $f_0 = 10^{-4} s^{-1}$, $\beta = 1.5 \times 10^{-11} m^{-1} s^{-1}$].

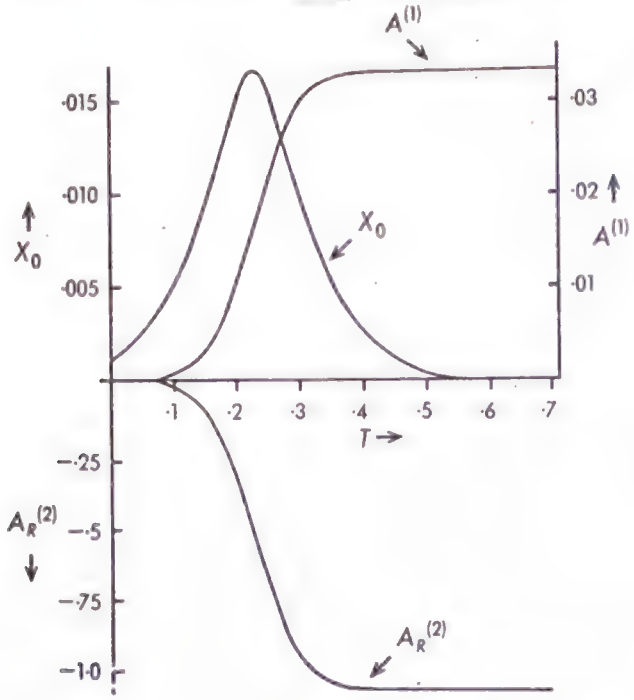
Figure 2.

explores the nature of the oscillations using analytic methods, whereas the

results shown here were obtained by integrating (26) numerically.

Further work on this topic has been reported by Pedlosky (1972a,b, 1977 and 1979) and Drazin (1970, 1972). A summary of current knowledge is given in Hart (1979).

Loesch (1978) has used the same approach for examining the stability of Rossby waves to small perturbations. Figure (3) shows that



Long time evolution of the perturbation amplitude $X_0(T)$ and the basic Rossby wave amplitude corrections $A^{(1)}(T)$ and $A_R^{(2)}(T)$ for $\beta=1$, $N=6$ and $l=1$; initial conditions are $X_0(0)=A_c$, $(dX_0/dT)_{T=0} = \sigma_i A_c$ and $A^{(1)}(0)=A_R^{(2)}0=0$.

After Loesch (1978)

Figure 3.

initially the perturbation grows in amplitude, corresponding to the instability deduced in linear studies, but that when non-linear effects become important the perturbation decreases, and the Rossby wave is 'stable', undergoing only a small change of phase and amplitude as a result of the encounter.

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Table of Symbols for Lecture 3

A	amplitude
a_1, a_2, b_1, b_2	arbitrary constants
B	constant used in (22)
C	constant used in (22)
c	phase speed
c_{oi}	constant defined in (27)
$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$	Jacobian of a and b
$T = \lvert\Delta\rvert^{1/2} t$	slow time
$X_2^{(2)}$	represents phase shift between levels at
U_c	critical wind shear at onset of instability
ϕ_n	perturbation stream function
Δ	excess of wind shear over critical value
α	zonal wave number
$m\pi$	meridional wave number
γ	ratio of amplitudes between layers for a critically stable wave.
$(\text{variable})^{(n)}$	correction to the variable at $O(\lvert\Delta\rvert^{n/2})$
r, β, F, k_m, γ_n, U_1, U_2	as in lecture 2.

LECTURE FOUR WAVE-WAVE INTERACTIONS

1. Wave Triads

In lecture 3 it was found that by examining the non-linear equations for a baroclinic instability model it was possible to determine how the developing wave altered the basic flow. If there were to be two large amplitude waves present it would be reasonable to expect that they would also interact with each other. This section will describe how this problem has been investigated for Rossby waves.

Consider two-dimensional, inviscid, incompressible flow on a sphere. The governing equations are then

$$\frac{\partial \underset{\sim}{v}}{\partial t} = - \underset{\sim}{\nabla}(p/\rho) - \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v} \tag{1}$$

and

$$\underset{\sim}{v} = - (\underset{\sim}{\nabla} \psi) \times \underset{\sim}{k} \tag{2}$$

By taking the scalar product of $\underset{\sim}{v}$ with (1), it is possible to show that the total kinetic energy is conserved. Thus

$$\int_S (\underset{\sim}{\nabla} \psi)^2 \, ds = \int_S \underset{\sim}{v} \cdot \underset{\sim}{v} \, ds = \text{constant} \tag{3}$$

Taking $\underset{\sim}{\nabla} \times$ (1) it is possible to eliminate p/ρ to get

$$\frac{\partial}{\partial t} \nabla^2 \psi = - (\underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \nabla^2 \psi \tag{4}$$

and on multiplying this by $\frac{1}{2} \nabla^2 \psi$ and integrating over the sphere gives the conservation of enstrophy (squared vorticity) in the form

$$\int_S (\nabla^2 \psi)^2 \, ds = \text{constant} . \tag{5}$$

It is possible to expand ψ in terms of the eigenfunctions of

$$\nabla^2 \psi_q + a_q \psi_q = 0 . \tag{6}$$

The appropriate form for a_q is, for a sphere of radius R and integer q

$$a_q = \frac{q(q+1)}{R}$$

This is analagous to the use of Fourier series to represent a function of one variable. Thus

$$\psi = \sum_{q=1}^{\infty} b_q \psi_q . \tag{7}$$

Then
$$\nabla^2\psi = \sum_{q=1}^{\infty} b_q \nabla^2\psi_q = -\sum_{q=1}^{\infty} b_q a_q \psi_q .$$

Using
$$\int_S (\underset{\sim}{\nabla}\psi)^2 ds = \int_S \underset{\sim}{\nabla}\cdot(\psi\underset{\sim}{\nabla}\psi)\,ds - \int_S \psi\nabla^2\psi\,ds$$
$$= -\int_S \psi\nabla^2\psi\,ds$$

the energy conservation equation becomes

$$\int_S (\underset{\sim}{\nabla}\psi)^2 ds = \sum_{q=1}^{\infty} \int_S \psi_q^2 a_q b_q^2 ds \qquad (8)$$
$$= \text{constant}$$

since the solutions to (6) are orthogonal. The enstrophy relation becomes

$$\int_S (\nabla^2\psi)^2 ds = \sum_{q=1}^{\infty} \int_S a_q^2 b_q^2 \psi_q^2 ds \qquad = \text{constant} \qquad (9)$$

Define
$$H_q = \int_S a_q b_q^2 \psi_q^2 ds$$

Then H_q represents the energy associated with the q^{th} eigenfunction.

Thus
$$\sum_{q=1}^{\infty} H_q \qquad = \text{constant} \qquad (10)$$

and
$$\sum_{q=1}^{\infty} a_q H_q \qquad = \text{constant} . \qquad (11)$$

An argument based on the two-dimensional counterpart of Sturm's oscillation theorem (Ince, 1956) shows that $1/a_q$ is a measure of the length-scale represented by ψ_q. Similarly the reciprocal of the wave number is a measure of the length-scale represented by a Fourier-component. Thus, by considering how H_q varies with q it is possible to determine the distribution of kinetic energy and enstrophy between length scales.

Proceed by writing

$$\Delta H_q = (H_q)_t - (H_q)_{t=0} \qquad (12)$$

Then
$$\sum_{q=1}^{\infty} \Delta H_q = \sum_{q=1}^{\infty} (H_q)_t - \sum_{q=1}^{\infty} (H_q)_{t=0} = 0 \qquad (13)$$

and
$$\sum_{q=1}^{\infty} a_q \Delta H_q = \sum_{q=1}^{\infty} a_q (H_q)_t - \sum_{q=1}^{\infty} a_q (H_q)_{t=0} = 0 \qquad (14)$$

Assume that an energy exchange occurs between two components $q_r = p, r$ with $r > p$ and that no other value of q_r is involved. Then

$$\Delta H_p + \Delta H_r = 0$$

and $$a_p \Delta H_p + a_r \Delta H_r = 0$$

and since $$a_p - a_r \neq 0 \quad , \quad \Delta H_p = \Delta H_r = 0.$$

This contradicts the assumption that H_p and H_r change.

Thus energy transfer must involve more than two length scales.

Suppose that three scales are involved in the changes: $q_r = p, r, s$. Then

$$\Delta H_p + \Delta H_r + \Delta H_s = 0$$

$$a_p \Delta H_p + a_r \Delta H_r + a_s \Delta H_s = 0 .$$

These have solutions

$$\Delta H_p = -\left(\frac{a_s - a_r}{a_s - a_p} \right) \Delta H_r$$

$$\Delta H_s = -\left(\frac{a_r - a_p}{a_s - a_p} \right) \Delta H_r$$

and if it is assumed that $p < r < s$

$$\frac{a_s - a_r}{a_s - a_p} > 0 \quad , \quad \frac{a_r - a_p}{a_s - a_p} > 0$$

so that the energy change of the intermediate length scale is of opposite sign to that of the other scales. In addition, the magnitude of the change of the intermediate scale is the greatest, and that of the smallest scale is least. This result extends to the case of more waves. (Fjørtoft, 1953). These results are summarised below, and are collectively known as Fjørtoft's Theorem.

1. At least three scales must be involved in energy transfer between waves.
2. No length-scale can act as an energy source or sink for two other scales unless it is intermediate to them.
3. The magnitude of the change is greatest for the intermediate length scale.

Note that these conclusions are valid for any system in which the total kinetic energy and enstrophy are conserved.

Newell (1969) discussed the interaction of triads of Rossby waves. Consider three Rossby waves

$$\psi_i = A_i \cos \left(\underset{\sim}{k}_i \cdot \underset{\sim}{x} - \omega_i t + \varepsilon_i \right) \qquad \text{for } i = 1, 2, 3 \qquad (15)$$

with $$\omega_i = k_i \left(U - \beta / |\underset{\sim}{k}_i|^2 \right) \qquad (16)$$

and ε_i is the phase of the i^{th} wave. The governing equation is

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi\right) + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0 \qquad (17)$$

so that when the Jacobian is evaluated product terms

$$A_i A_j \cos\left\{ \left(\underset{\sim}{k}_i \pm \underset{\sim}{k}_j \right).\underset{\sim}{x} - (\omega_i \pm \omega_j) t + (\varepsilon_i \pm \varepsilon_j) \right\}$$

are formed. Resonance may then occur if

$$\underset{\sim}{k}_1 + \underset{\sim}{k}_2 + \underset{\sim}{k}_3 = \underset{\sim}{0}$$

and $$\omega_1 + \omega_2 + \omega_3 = 0$$

for then any of the product waves is itself a member of the triad, since

$$\underset{\sim}{k}_i + \underset{\sim}{k}_j = - \underset{\sim}{k}_k \qquad (i,j,k \text{ different})$$

and $$\omega_i + \omega_j = - \omega_k \qquad (i,j,k \text{ different})$$

and $$\cos(a) = \cos(-a)$$

By assuming that the waves were of small but finite amplitude, Newell (1969) expanded the equation in a way similar to that of Pedlosky (1970) described in lecture 3. He showed that such interactions, with one of the wave numbers zero, are capable of accelerating the mean flow by up to 1.9 ms^{-1} per day.

Drazin (1970) briefly discusses the application of his finite amplitude solution of the Eady problem to wave interactions. He showed that if the waves were both close to the neutral stability curve the resulting flow would vacillate between the two waves. Pedlosky (1975) has shown that a baroclinic wave which is of small meridional wave number, and which according to linear theory can grow exponentially, may be stabilised by the presence of two stable waves.

2. Baroclinic Wave Packets

By supposing that the amplitude function A of the previous lecture is a function of both T, a slow time, and $X = |\Delta|^{\frac{1}{2}} x$, a long space variable, Pedlosky (1972) considers the evolution of wave packets. He argues that in practice if the wind shear is increased beyond its minimum value for instability all unstable waves develop, not only the most unstable. This results in a band of unstable waves. For the two-layer model considered previously, the value of this critical shear, U_c, is β/F and corresponds to a wavelength for unstable waves of total wave-number $(\sqrt{2}F)^{\frac{1}{2}}$. Figure 1 shows that increasing the shear by Δ destabilises a band of wavenumbers of width $\Delta^{1/4}$. Considering only those shears which differ from the minimum critical shear U_c by a small amount, Δ, Pedlosky repeats the previous analysis, and determines the following pair of equations which govern the evolution of the amplitude of the wave packet.

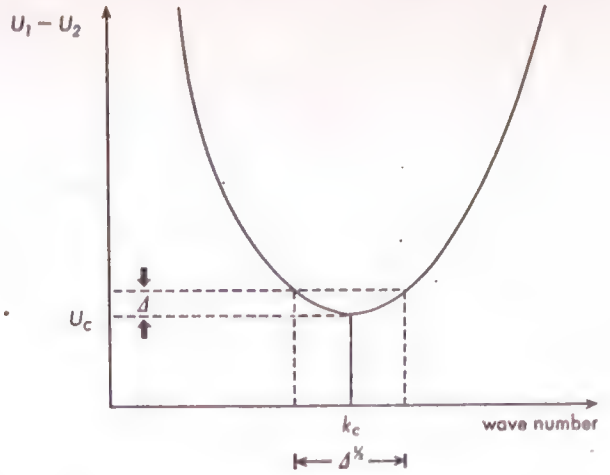

Diagram to show how increasing the shear $(U_1 - U_2)$ of the basic state in the two level model by Δ de-stabilises a band of waves around the most unstable wavenumber (k_c).

Figure 1.

$$\left(\frac{\partial}{\partial T} + c_{g_1}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T} + c_{g_2}\frac{\partial}{\partial X}\right)A = \sigma^2 A - NAB \qquad (1)$$

$$\left(\frac{\partial}{\partial T} + c_{g_2}\frac{\partial}{\partial X}\right)B = \left(\frac{\partial}{\partial T} + c_{g_1}\frac{\partial}{\partial X}\right)|A|^2 \qquad (2)$$

where $N = \frac{k^2\gamma^2 m^2 \pi^2}{4}$, the symbols taking the meanings ascribed to them in the table of symbols.

If $\frac{\partial}{\partial X}$ is identically zero, these reduce to the amplitude equation derived in lecture 3.

The c_{g_i} are the two group velocities corresponding to the two roots for the phase speed, c_{g_1} being associated with the negative sign for the square root and c_{g_2} with the positive sign. For the most unstable wave at the minimum critical shear

$$c = U_2 + \frac{\beta(k_m^4 - 2F^2)}{2k_m^2 F(k_m^2 + 2F)}\left(1 \pm 1\right)$$

which has the double root

$$c = U_2$$

Taking the limit as $k_m^2 \to \sqrt{2}F$ the group velocities are $\frac{\partial \sigma k}{\partial k}$ which takes the values

$$c_{g_1} = U_2 + \frac{\beta k^2(1+\gamma^2)}{F^2} \qquad (3)$$

and $c_{g_2} = U_2$.

Gibbon et al (1979) have shown that, with the boundary conditions $A, B \to 0$ as $|X| \to \infty$, these equations may be transformed to

$$\phi_{\xi\tau} = \pm \sin\phi \qquad (4)$$

the sine-Gordon equation. This equation has been studied extensively in

other contexts, and it has been shown to have soliton solutions. Solitons are discussed in the review papers by Scott et al (1973) and Rebbi (1979). Their properties relevant to this study are that they are exact solutions to a non-linear equation such that if two solitons interact they emerge unchanged except for a possible phase shift, and that they maintain their form indefinately. They exist by a balance between the instability mechanism which would make a linear wave grow and the non-linear 'dissipative' terms.

Gibbon et al (1979) present the following argument. Assume that the behaviour of the fluid in the two level model in some way models the atmosphere, so that a solution to the model problem may be applied to the atmosphere. Also assume that the small-but-finite amplitude assumptions present in the derivation of (1) and (2) may be relaxed to permit large amplitudes. The conclusion to be drawn from these hypotheses is that solitons made up of baroclinic waves are to be found in the atmosphere. If this is the case, then the long-term stability and constant phase speeds of solitons mean that they are predictable over long periods of time. (Although analytic solitons are infinitely predictable, 'real' ones would be exposed to external influences which would tend to destroy them over long periods). This has great implications for the predictability of the atmosphere, for if such solitons exist, the atmosphere is intrinsically predictable for long periods of time.

The assumptions made by Pedlosky (1970) are more appropriate to the oceans than to the atmosphere. The evidence for the natural occurance of baroclinic solitons should thus be first sought in the seas. The simplest environment in which to investigate the soliton behaviour of baroclinic waves is the laboratory, and experiments in a rotating annulus of sufficient circumference to appear 'infinitely long' to local disturbances would be the simplest starting point.

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Table of symbols used in Lecture 4

1. Wave Triads

U	zonal velocity
$\underset{\sim}{v} = (u, v, 0)$	2-D velocity
$\underset{\sim}{k}$	Vertical unit vector
ψ	stream function
p	pressure
ρ	density
R	radius of sphere
$\int_S ds$	surface integral over sphere.
q	integer
$a_q = \frac{q(q+1)}{R}$	eigen value
ψ_q	eigen vector } of $\nabla^2 \psi_q + a_q \psi_q = 0$.
b_q	coefficients of ψ_q in series for ψ,
H_q	energy corresponding to ψ_q.
ΔH_q	change of energy of component ψ_q.
ε_i	phase of wave i.
$\underset{\sim}{k}_i = (k_i, l_i, m_i)$	wave vector of wave i.

2. Baroclinic Wave Packets

$A(X,T)$	amplitude of wave
$B(X,T)$	measure of modification of basic state
$T = \lvert\Delta\rvert^{\frac{1}{2}} t$	slow time.
$X = \lvert\Delta\rvert^{\frac{1}{2}} x$	long space variable.
U_c	minimum vertical shear for instability.
U_1, U_2, β, F	as in lecture 2
c_{g_1}, c_{g_2}	group velocities corresponding to the two roots of the quadratic for phase speed.
$\sigma^2 = \frac{\Delta}{2\lvert\Delta\rvert} k^2 \gamma^2 \frac{\beta}{F}$	linear growth rate squared.

$N = \frac{k^2 \gamma^2 m^2 \pi^2}{4}$	constant
k	zonal wave number
$m\pi$	meridional wave number
Δ	increment of shear above
γ	amplitude ratio between levels of neutral wave as defined in lecture 3.

LECTURE FIVE NUMERICAL INVESTIGATIONS OF BAROCLINIC INSTABILITY

Unlike many branches of science, meteorology deals with phenomena which are not easily made the subject of controlled experiments. Instead, experiments are performed using analogues to the atmosphere and the results of these used in the development of theories. The rotating annuli of Hide (1977) and Hart (1976) are examples of such models, as are the numerical models described in Gilchrist (1979) and Hoskins and Simmons (1975). Numerical models, provided that the assumptions used in deriving them are justified, may be used to simulate the behaviour of the atmosphere in an attempt to understand the mechanisms involved on the motions. Such models have the advantage that they may be as simple (Hoskins, 1973) or as complex (Corby et al, 1977) as desired.

Hoskins and Simmons (from now on referred to as H & S) have used the spectral model described in H & S (1975) to investigate baroclinic waves. The technique used is to initialise the model with a given field consisting of a basic state with perturbations derived from a linearised problem superposed upon it. In H & S (1976) the solutions sought are normal modes, so that the perturbation is repeated n times around the globe. Later (H & S, 1979) the development of isolated perturbations was considered. The use of numerical models enables the energy of a perturbation and other diagnostic quantities to be calculated accurately, whereas this is a difficult task for atmospheric case studies.

The technique used by H & S for obtaining the linear solution is of interest. They took their non-linear model and linearised the equations about a basic state. A perturbation was used as initial conditions and the model integrated in time until the growth of the perturbation was exponential. The resulting approximation was thus a close approximation to the fastest growing mode, and the growth rate of this was also the growth rate of the perturbation (H & S, 1977). This technique allowed for larger models to be used than the eigenvalue technique of Foreman (1978). Before the perturbations were used to initialise the non-linear model they were normalised to give a maximum pressure perturbation of 1 mb.

The first experiments to be performed were to determine the linear normal mode response of a non-linear spectral model with eight levels in the vertical. By using only those spectral components with wavenumbers multiples of that under consideration it was possible to derive such a response. The most unstable modes of a given wave number for given vertical and horizontal profiles for the basic state were investigated. For a jet at 30°N, the most unstable mode was at wavenumber 8 or 9, with an e-folding time of 1.2 days. The variations of growth rate and phase speed with wave number are shown in fig 1. The structure of the most unstable wave is shown in figure 2 . It can be seen that, at lower levels, the perturbation resemble those of the Eady problem. The abrupt change in slope of the temperature perturbation in figure 2 above $\sigma = 0.580$ is due to the cooling (or warming) by vertical motions becoming larger than the warming (or cooling) due to the meridional motion. It is a consequence of the profiles chosen for the basic state (Fig 3).

The heat fluxes and other diagnostics may be taken from the model more conveniently than from observations. A selection of diagnostics is shown in figures 4 and 5.

Most theoretical studies of baroclinic waves have used a normal mode approach. The atmosphere, although it has some features which may be expected to be of this form, is not symmetrical about its axis. Accordingly H & S (1979) investigates the development of an isolated disturbance. This is assumed to be a localised positive vorticity maximum with balanced temperature and pressure perturbations. The surface pressure anomaly is this time normalised to $\frac{1}{2}$ mb. This state is

shown in figure 6, together with the fields for forecasts for 7, 12 and 17 days. It can be seen that the parent low advects downstream and that further lows develop downstream of it. Upstream smaller lows are formed, always at the same longitude. By day 17 the downstream development interferes with the upstream development. Figure 7 shows the $\sigma = 0.9$ temperature field at the end of the simulation. It can be seen that the waves have destroyed the initially sharp temperature gradient. The upper level perturbation shows little sign of the upstream developments but the downstream development is noticeable. The growth rate of the downstream development is initially faster than that given by a normal mode approach and is of a smaller scale than the most unstable normal mode. The 'wavelength' of the downstream lows increases as they mature.

Using an analytic solution to the quasi-geostrophic equations for the basic state (Eady, 1949) it was found that similar behaviour is observed (H & S, 1979). In particular, the upstream developments grow from the surface and those downstream grow from the top of the model (figure 8).

Acknowledgment

I would like to thank Dr B J Hoskins for supplying the diagrams used in this lecture.

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Figure captions for lecture five.

Figure 1. Dependence of growth rate and phase speed on zonal wavenumber for the 30° jet: fastest growing primitive equation modes (x), fastest growing quasi-geostrophic modes (.), and second fastest growing primitive equation modes (+).

Figure 2. Structure of the most unstable mode at wavenumber 8 for the 30° jet.
(a) perturbation stream functions at $\sigma = 0.970$ (solid contours), and $\sigma = 0.437$ (broken contours).
(b) surface pressure (solid contours) at intervals of 4 mb, and low level temperature ($\sigma = 0.970$, broken contours) at intervals of 5°C.
(c) low level vertical velocity at $\sigma = 0.970$ (solid contours) at intervals of $\frac{1}{2}$ mb h^{-1}, and temperature as in (b).

Figure 3. A Meridional cross-sections of basic states used in H&S (1976). Zonal winds (solid lines) and temperatures (dotted lines).
(a) 30° jet
(b) 55° jet
(c) solid body rotation.
Intervals of 5 ms^{-1} and 5°C.

B Meridional cross-sections for the 30° jet.
(a) static stability ($10^{-1} s^{-2}$, solid lines) and absolute vorticity ($10^{-4} s^{-1}$, dotted line).
(b) Ertel potential vorticity ($10^{-5} s^{-1}$, solid lines) and potential temperature (K, dotted line).

Figure 4. A Vertical structure of most unstable mode of the primitive equations at wavenumber 8, plotted at the latitude at which the meridional velocity has maximum amplitude.
B Stream functions of unstable primitive equation modes for 30° jet at $\sigma = 0.970$ (solid contours) and $\sigma = 0.437$ (broken contours).
(a) fastest growing mode at wave number 4.
(b) fastest growing mode at wavenumber 12.
(c) second fastest growing mode at wavenumber 8.

C Meridional cross-sections of fluxes due to the most unstable primitive equation disturbance to the 30° jet.
(a) northward eddy heat flux $\overline{V'T'}$
(b) vertical eddy heat flux $\overline{\omega'T'}\cos\theta$.

D Net rate of change of zonal-mean temperature for the most unstable disturbance to the 30° jet.
(a) primitive equations.
(b) quasi-geostrophic equtions.

Figure 5. Most unstable primitive equation modes at wavenumber 8 for (a) the 30° jet, (b) the 55° jet, (c) the case of solid body rotation.
A Northward eddy flux of relative angular momentum $\overline{U'V'}$.
B Vertical eddy flux of relative angular momentum $\overline{\omega'U'}\cos\theta$.
C Stream functions of the induced mean meridional circulation.
D Net rate of change of zonally-meaned zonal velocity.

Figure 6. Surface pressure for asymmetric simulation.
(a) Initial state.
(b) Day 7.
(c) Day 12.
(d) Day 17.

Figure 7. Temperature field at σ = 0.9 on day 17 for the asymmetric simulation.

Figure 8. Sections showing the development of an isolated Eady mode in time.

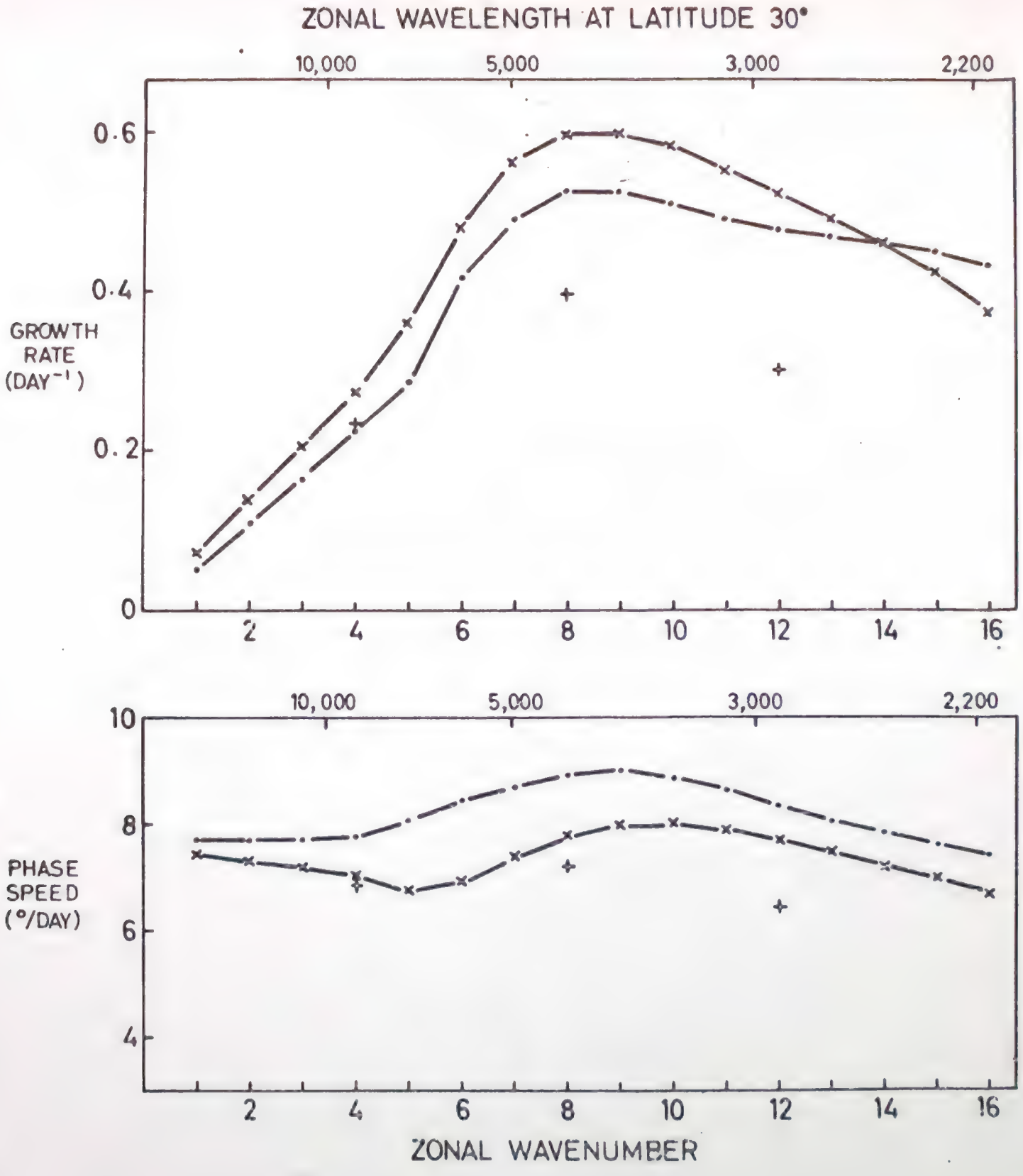

ZONAL WAVELENGTH AT LATITUDE 30°
10,000
5,000
3,000
2,200
0.6
0.4
0.2
0
GROWTH RATE (DAY⁻¹)
2
4
6
8
10
12
14
16
10,000
5,000
3,000
2,200
10
8
6
4
PHASE SPEED (°/DAY)
2
4
6
8
10
12
14
16
ZONAL WAVENUMBER

Figure 1

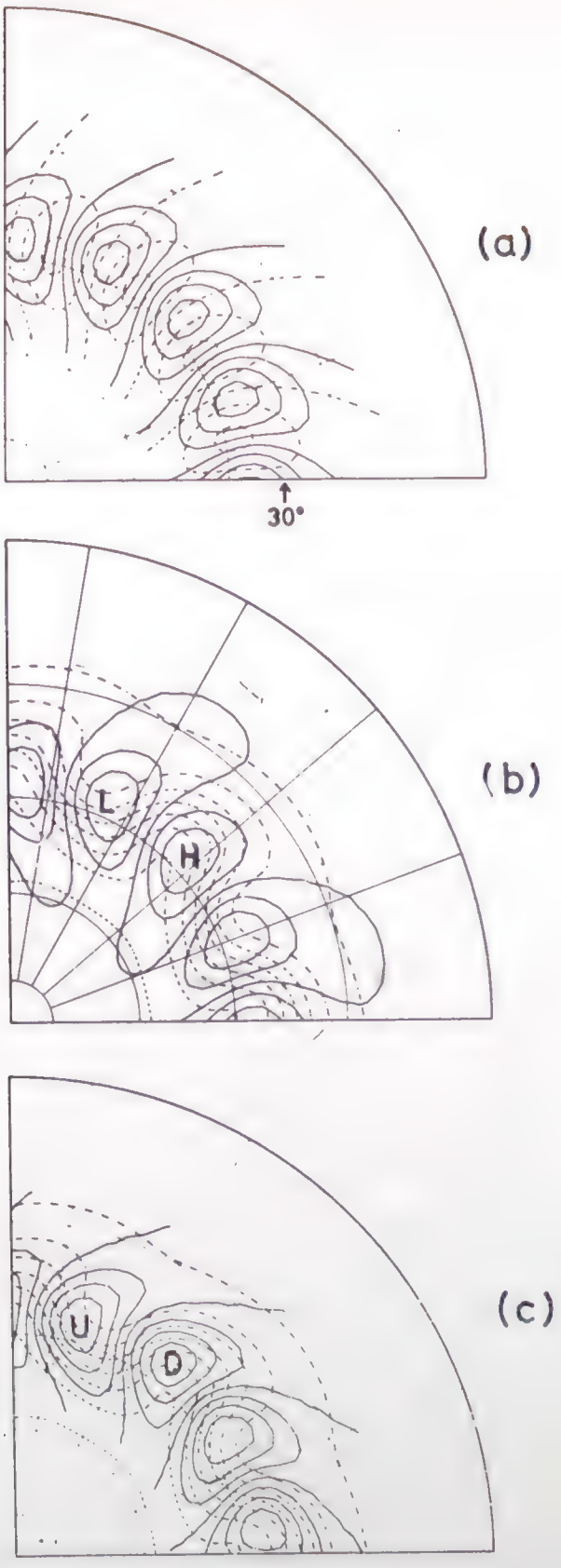


Figure 2

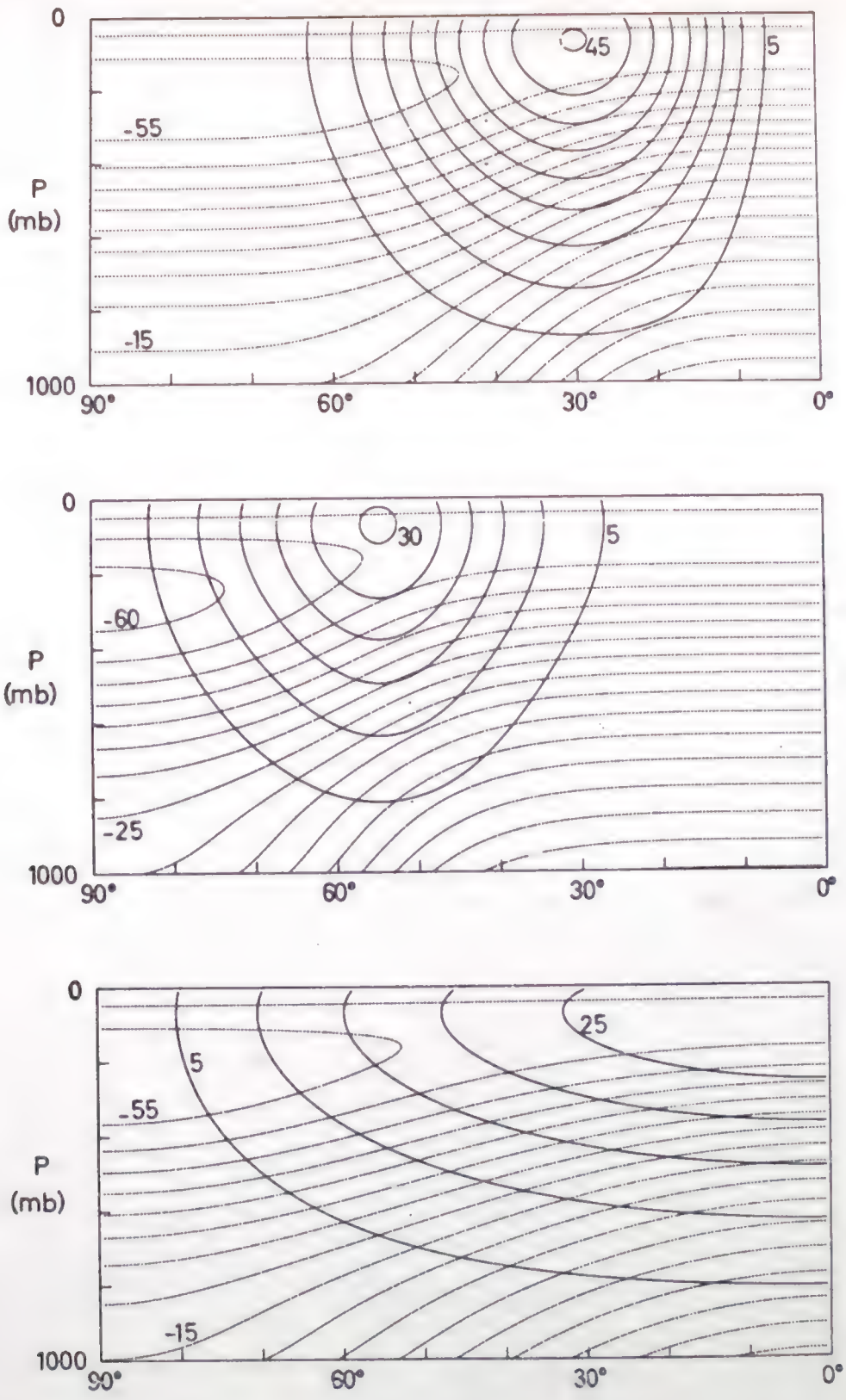
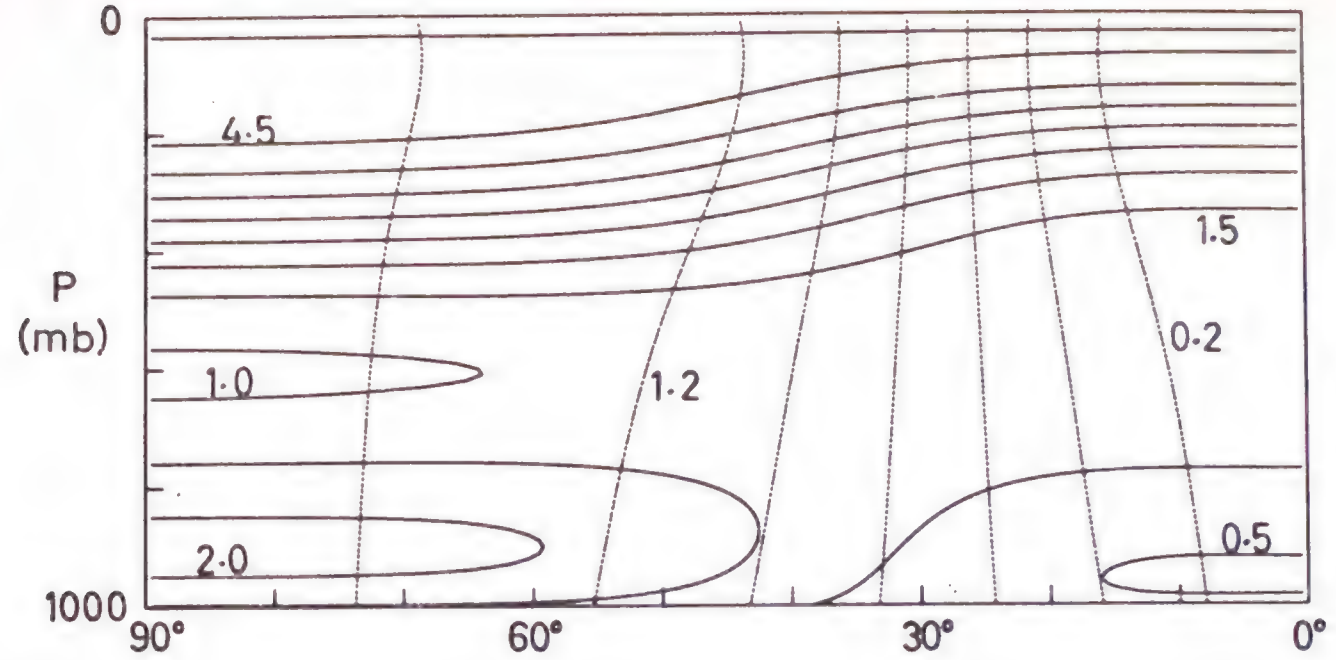
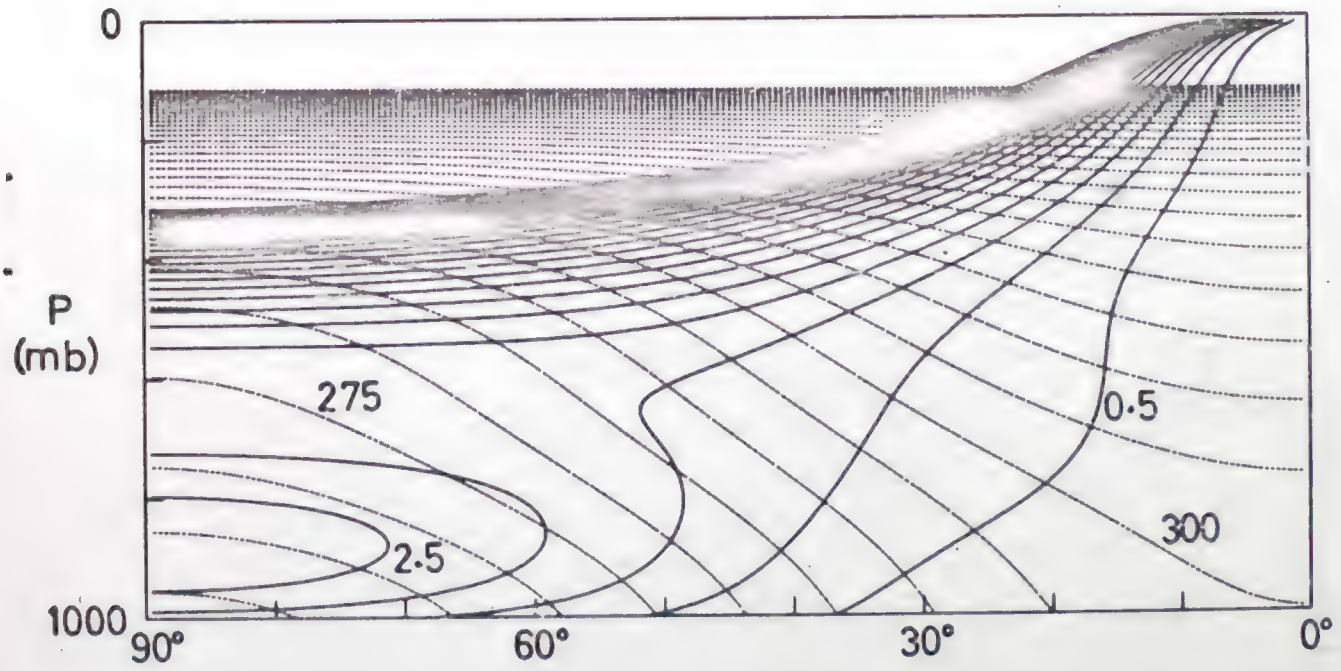


Figure 3 A

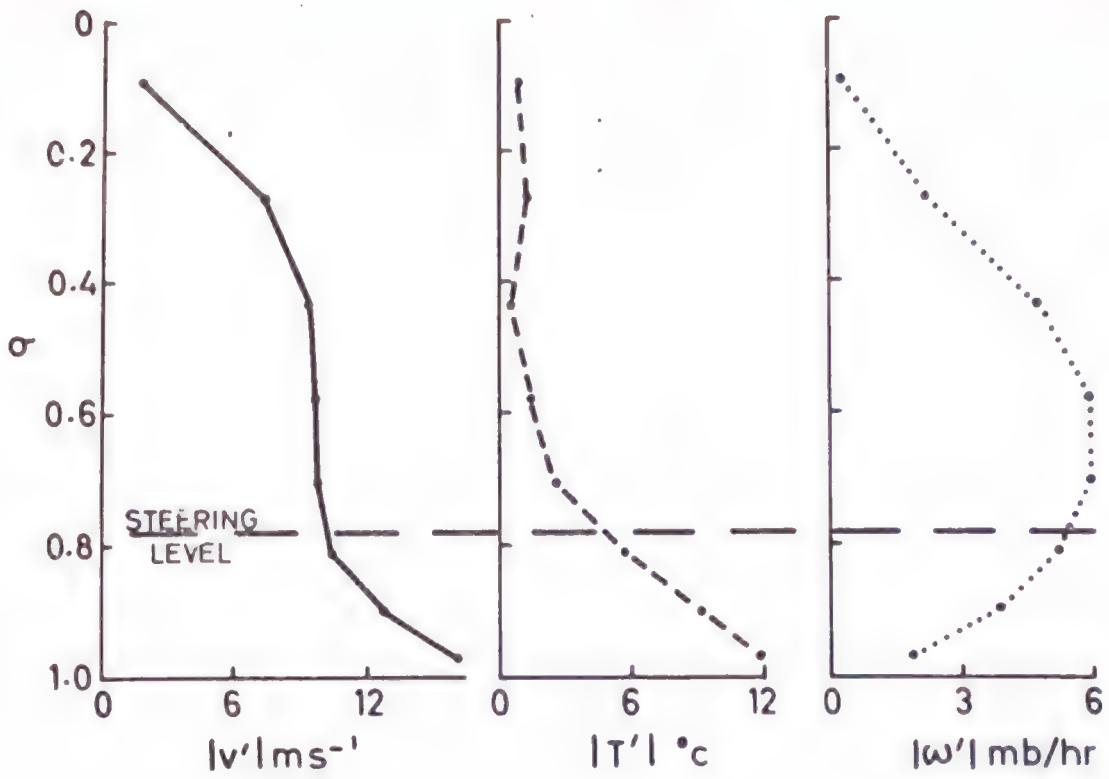

0
4.5
P
(mb)
1.0
1.2
1.5
0.2
2.0
0.5
1000
90°
60°
30°
0°

(a)


0
P
(mb)
275
0.5
2.5
300
1000
90°
60°
30°
0°

(b)

Figure 3 8


0
0.2
0.4
0.6
0.8
1.0
σ
STEERING LEVEL
0
6
12
|v'|ms⁻¹
0
6
12
|T'| °c
0
3
6
|ω'| mb/hr

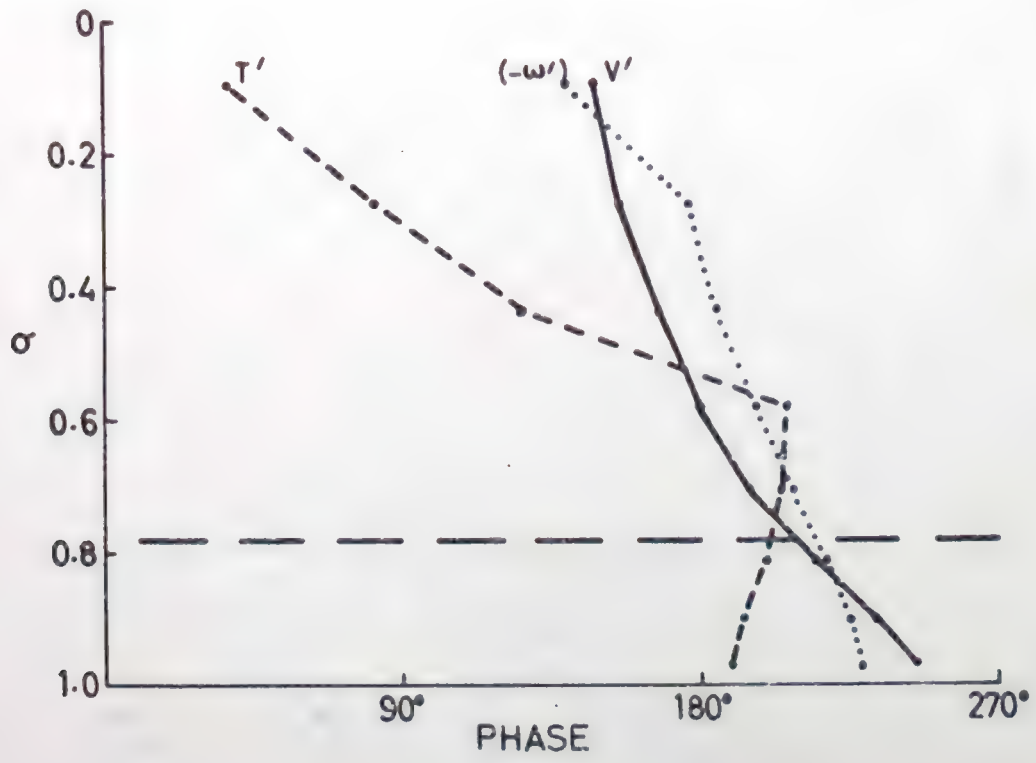
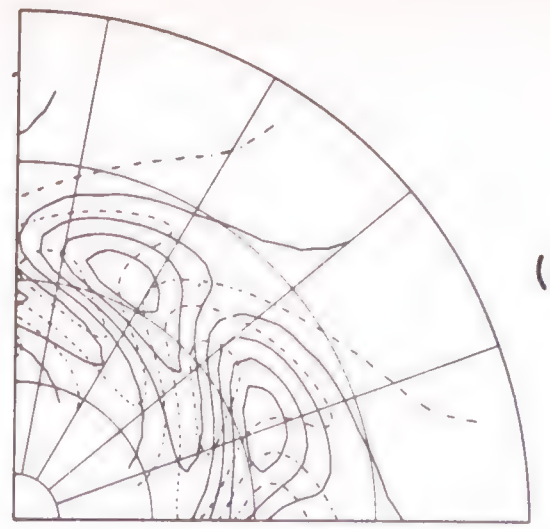
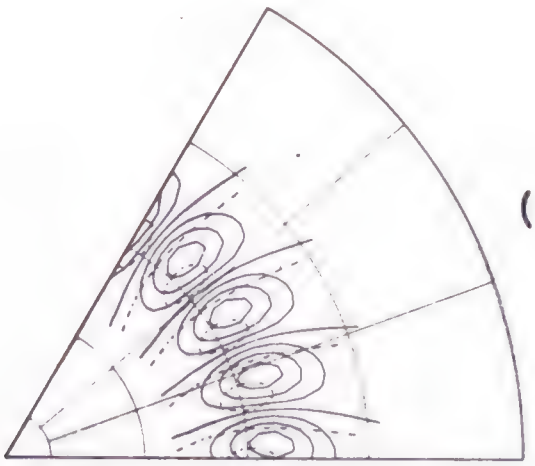

0
0.2
0.4
0.6
0.8
1.0
σ
T'
(-ω')
V'
90°
180°
270°
PHASE

Figure 4 A



(a)



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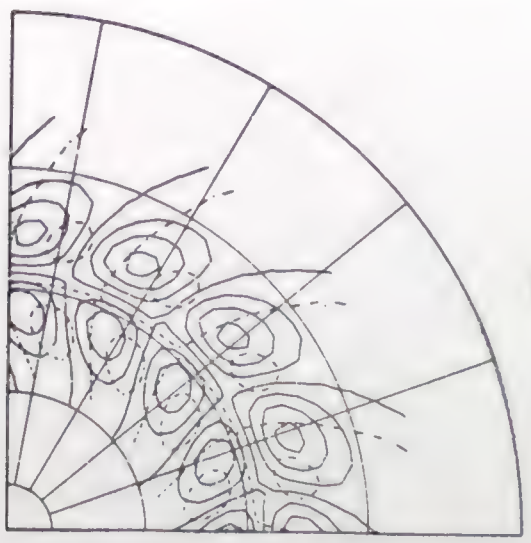
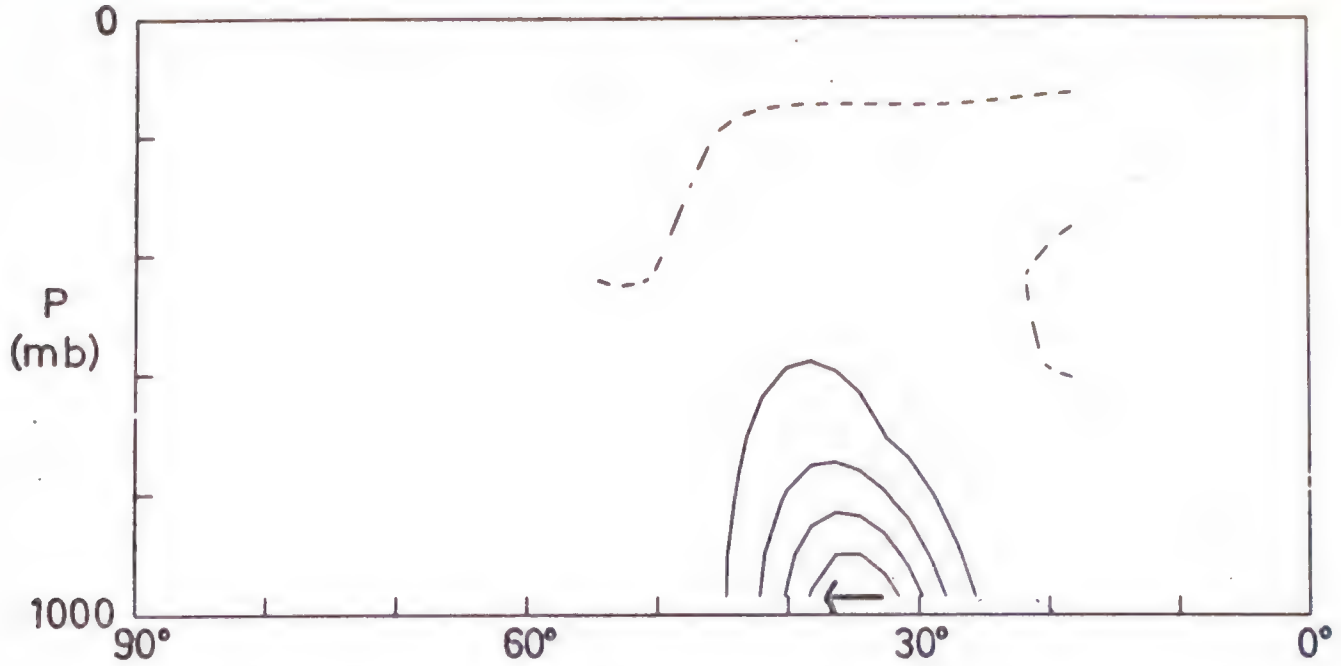


(c)

Figure 4 B



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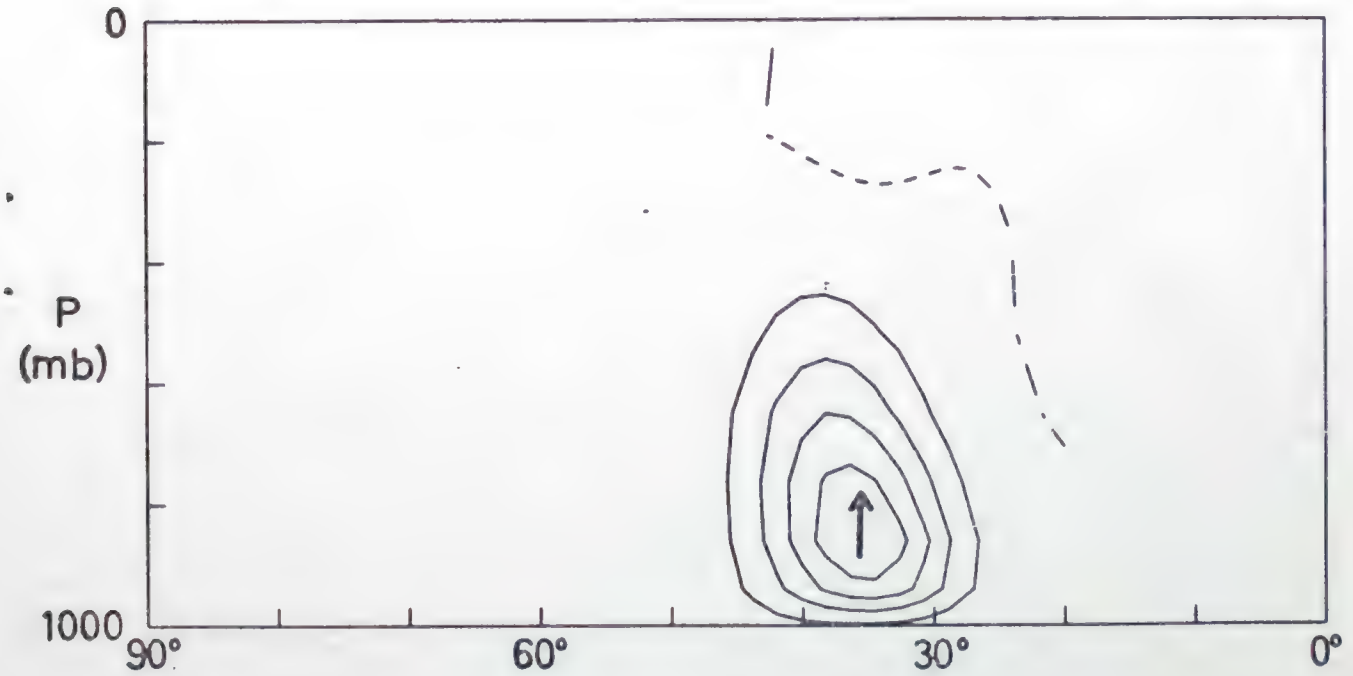
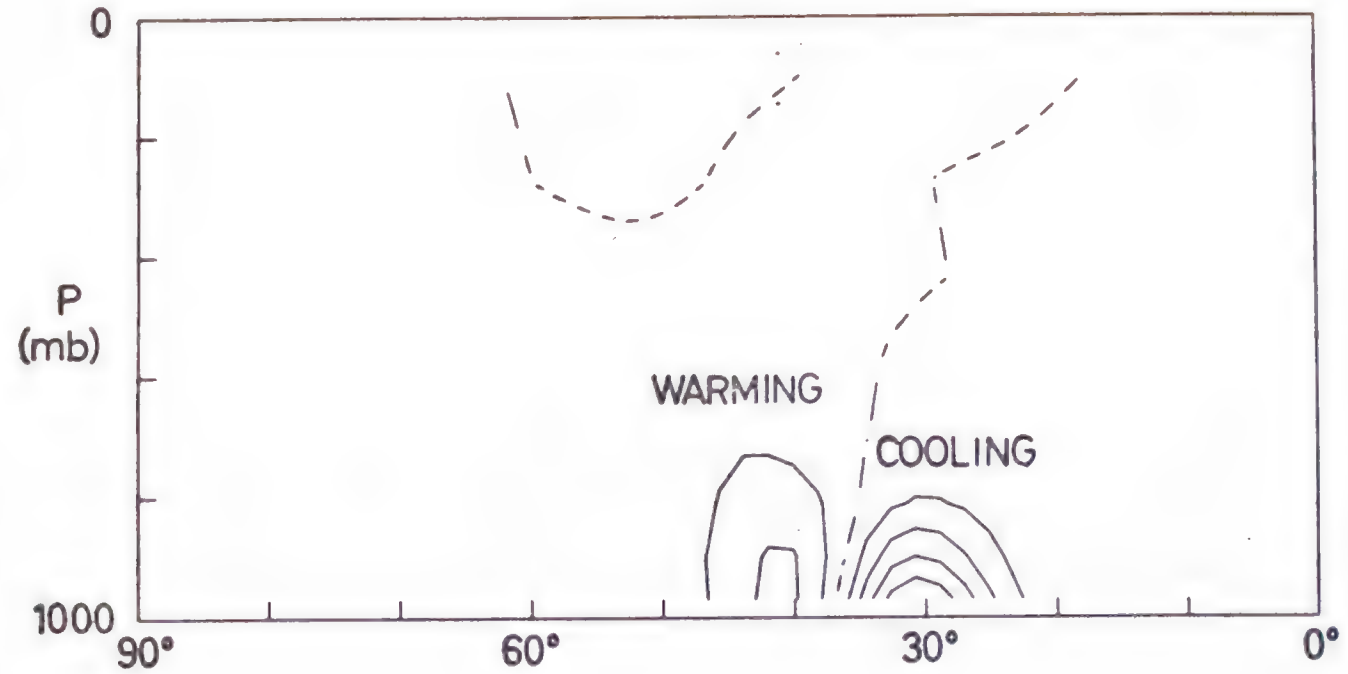
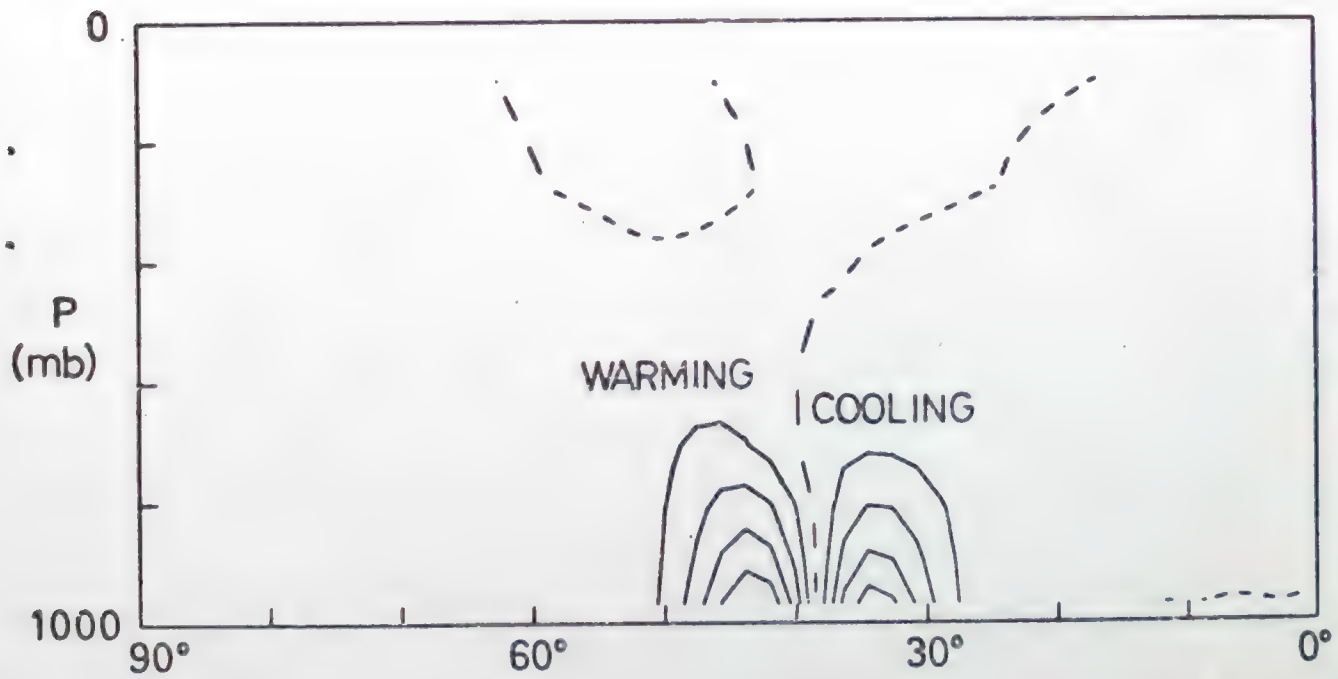


(b)

Figure 4 C


0
P
(mb)
1000
90°
60°
30°
0°
WARMING
COOLING

(a)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WARMING
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(b)

Figure 4 D

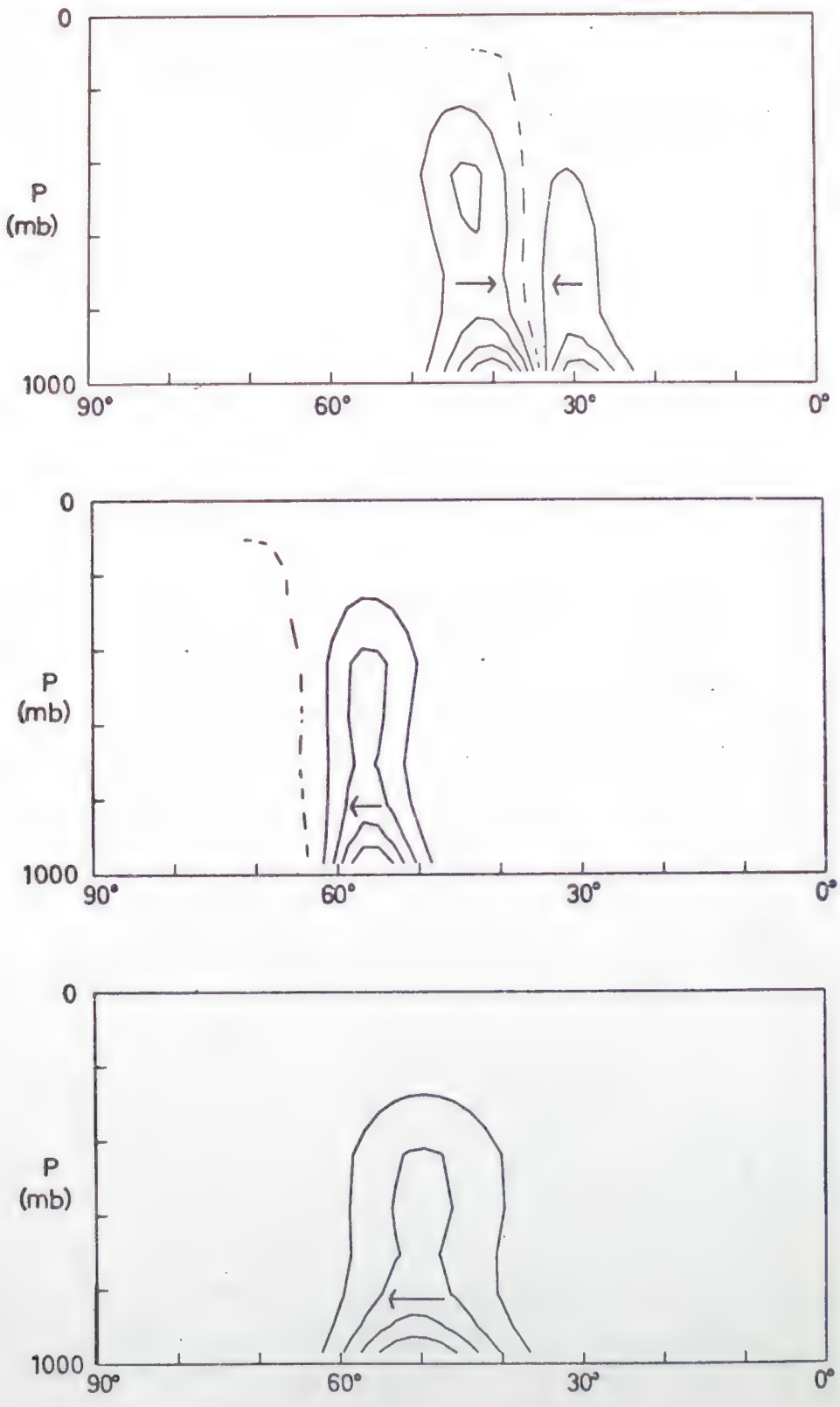


Figure 5 A

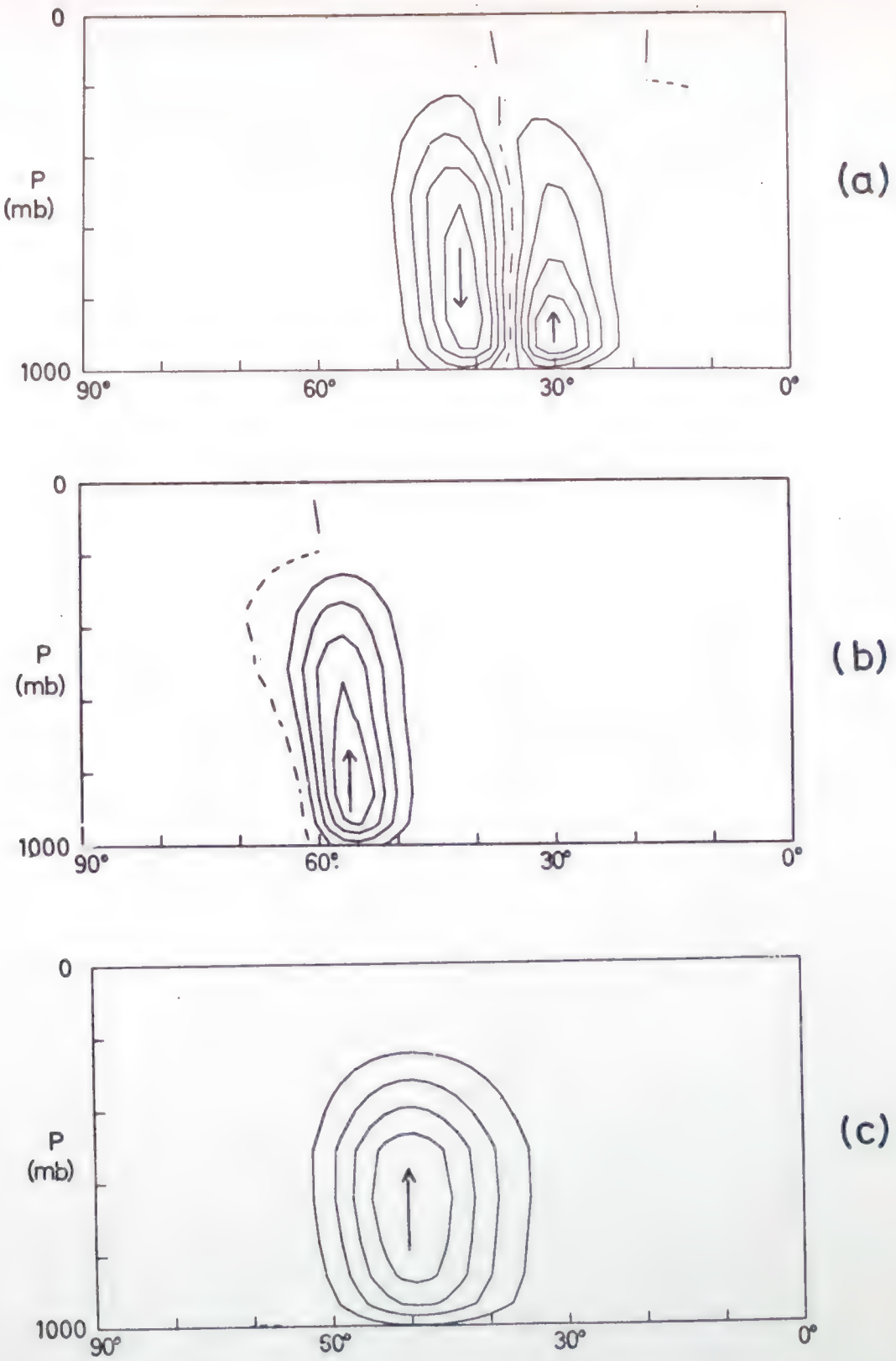


Figure 5 B

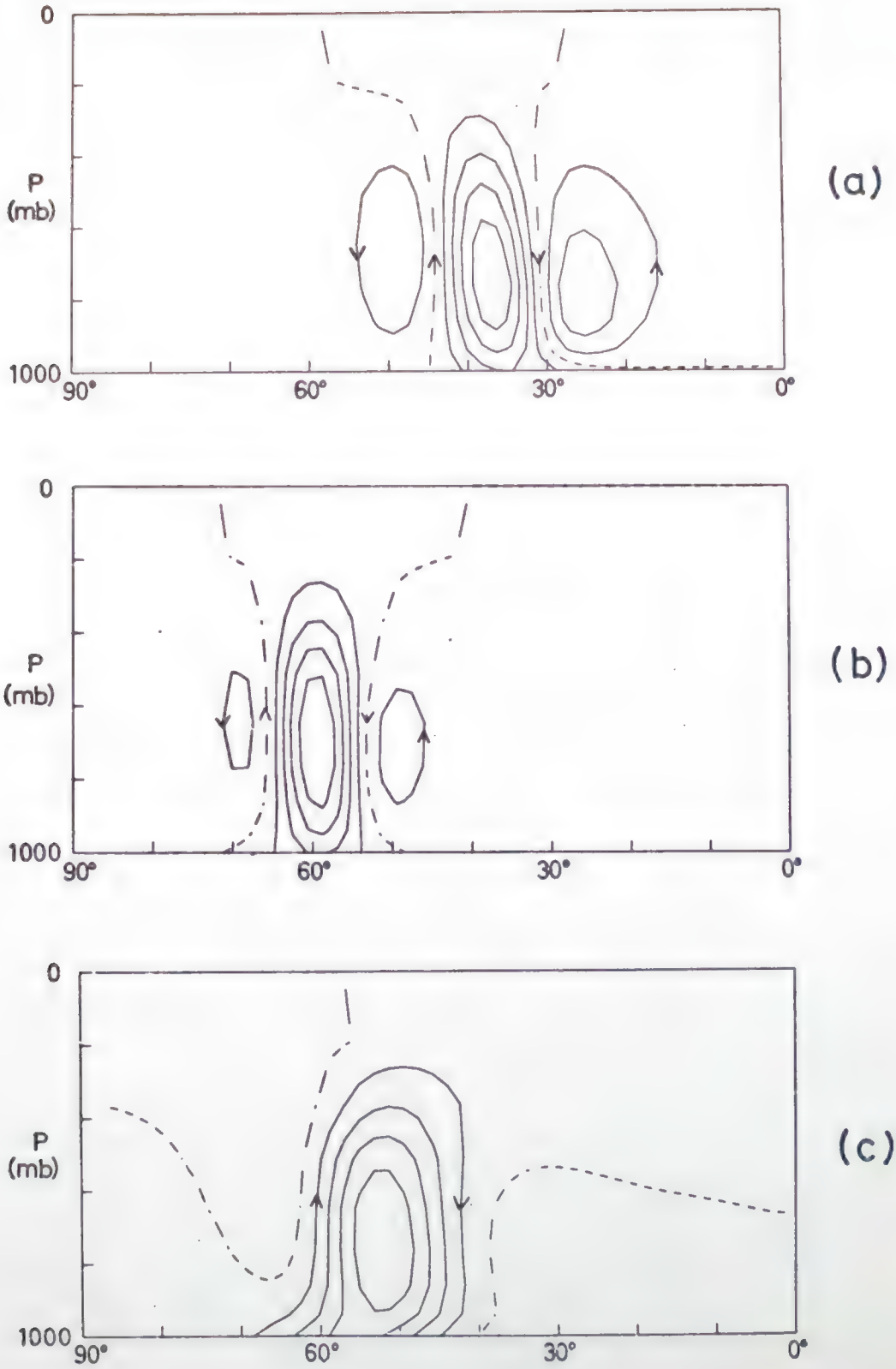
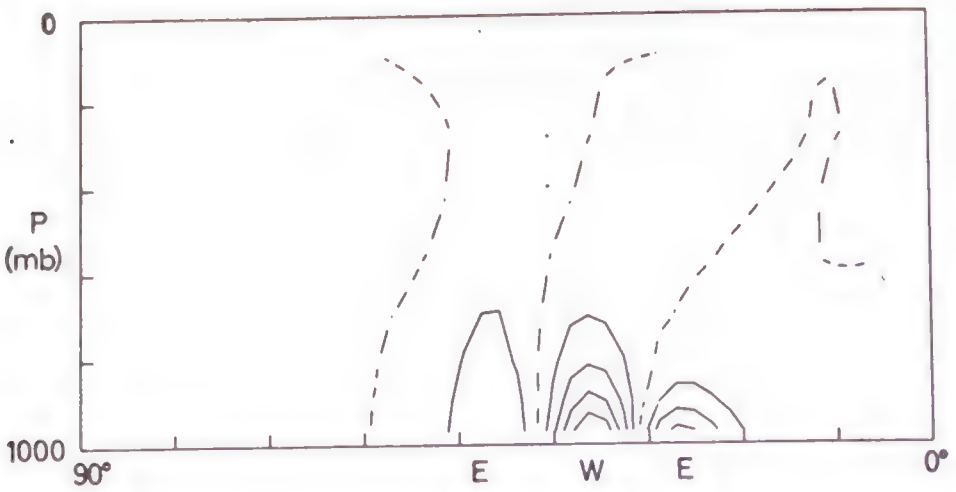
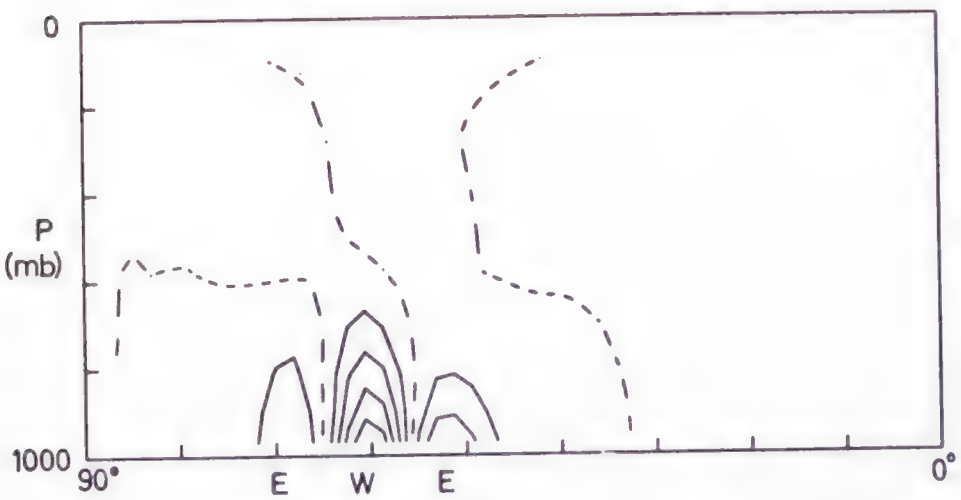


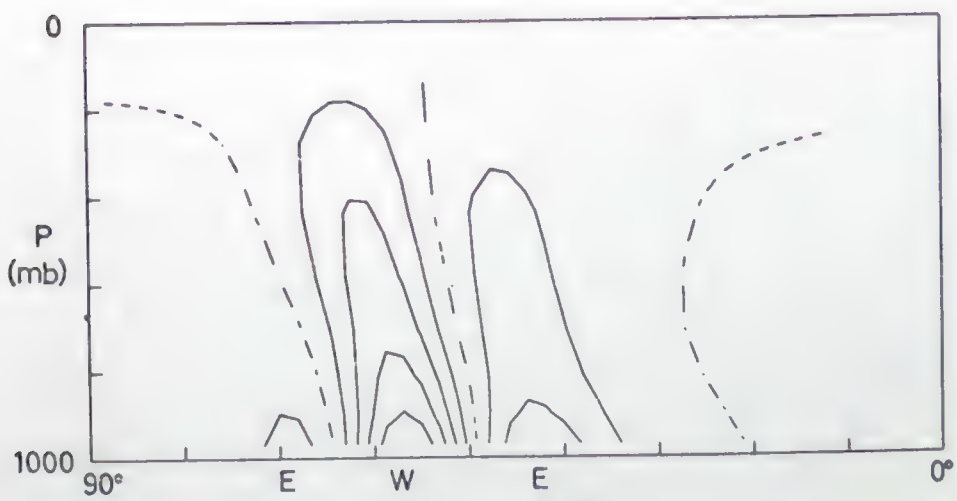
Figure 5 C



(a)

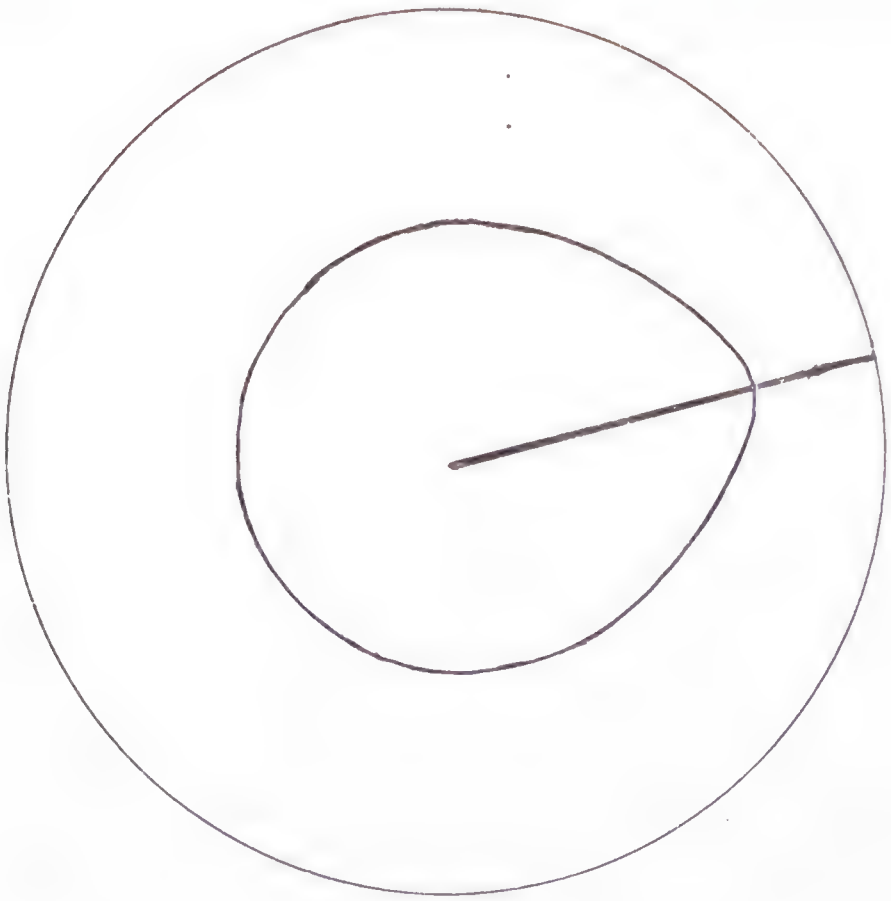


(b)



(c)

Figure 5 D



Surface pressure pattern for asymmetric simulation.

Figure 6(a).

Figure 6(b)

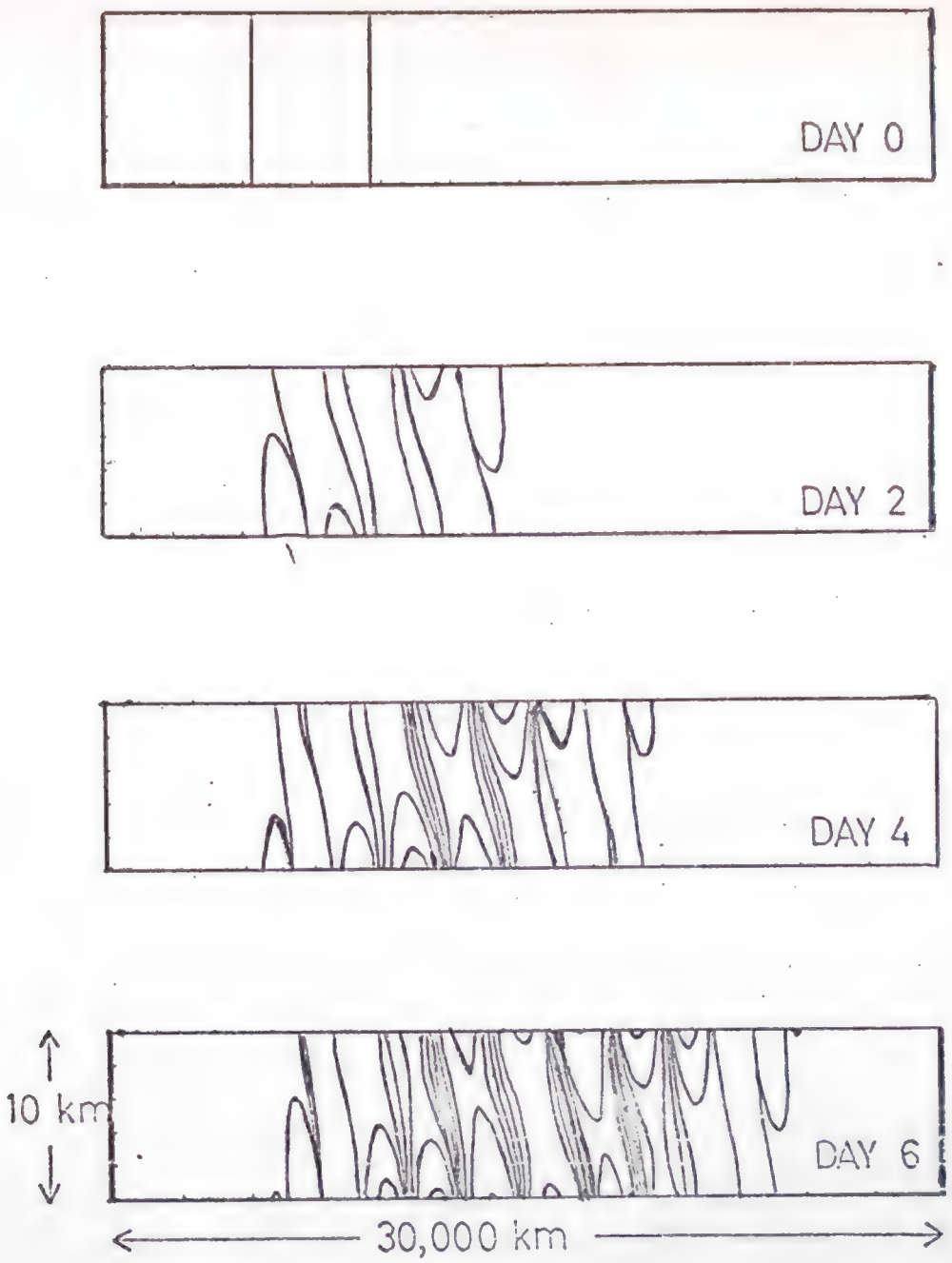
Figure 6(c)

SURFACE PRESSURE AT DAY 17.00

Figure 6(d)

Temperature field for asymmetric simulation.

Figure 7



Height/longitude sections showing the relative vorticity at days 0, 2, 4 and 6 for the Eady basic state. Contours are drawn for values ± 0.01, ± 1, ± 2.5, ± 5, ± 10 and ± 25. The zero contour is not drawn in order to avoid the illustration of small-scale variability of negligible amplitude.

Figure 8.

LECTURE SIX LINEAR STUDIES OF LEE WAVES

A stably stratified atmosphere can support waves due to buoyancy. One such family of waves is known as 'lee waves'. These are formed by the vertical displacement of air due to the presence of mountains. The waves are stationary with respect to the mountain so that the time-independent equations of motion may be used when seeking steady state waves. Further assumptions which simplify the theory are neglecting the Coriolis effect and taking the mountain to be of infinite length. The x-axis is taken to be normal to the mountain range, the y-axis to lie along it and the z-axis to be vertical. Linearising the equations of motion gives

$$u_0 \frac{\partial u}{\partial x} + w \frac{\partial u_0}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \tag{1}$$

$$u_0 \frac{\partial w}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{g \rho}{\rho_0} \tag{2}$$

$$u_0 \frac{\partial \rho}{\partial x} + w \frac{\partial \rho_0}{\partial z} = -\rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \tag{3}$$

$$u_0 \frac{\partial p}{\partial x} + w \frac{\partial p_0}{\partial z} = c_0^2 \left(u_0 \frac{\partial \rho}{\partial x} + w \frac{\partial \rho_0}{\partial z} \right) \tag{4}$$

$$p = \rho_0 R T + \rho R T_0 \tag{5}$$

where the subscript denotes a basic state variable.
It is possible to eliminate u, ρ and p from the equations to give

$$M_0 w_{1xx} + w_{1zz} + F(z) w_1 = 0 \tag{6}$$

where $$F(z) = \frac{\beta_0 g}{u_0^2} + S_0 \frac{u_0'}{u_0} - \frac{1}{4} S_0^2 + \frac{1}{2} S_0' - \frac{u_0''}{u_0} \tag{7}$$

and the variables are defined in the table of symbols.

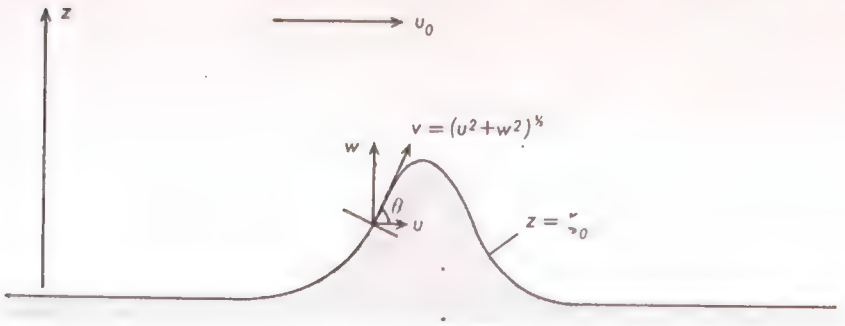
If the density of the air is assumed to be constant and the wind uniform then

$$F(z) \sim N^2 / u_0^2$$

where N is the Brunt-Väisälä frequency, the maximum frequency of oscillation of vertical gravity waves (Lighthill, 1978). Thus F(z) represents the stability of the atmosphere, taking into account the temperature, density and wind structure. It is the inverse square of a length scale which may be thought of as the horizontal distance travelled in the time taken for a vertical oscillation.

Equation (6) requires boundary conditions to be specified before it can be solved. One of these is derived from the requirement that the surface velocity should be tangential to the surface. From figure (1) it can be seen that this requirement is equivalent to

$$\frac{d\zeta_s}{dx} = \frac{w}{u + u_0} \approx \frac{w_1(x,0)}{u_s} \tag{8}$$



Definition sketch for the lower boundary condition for lee waves.

Figure 1.

for sufficiently small ζ_s and $\frac{\partial \zeta_s}{\partial x}$. Further requirements are derived from consideration of the flow at infinity. The most obvious of these is that the energy of the waves should tend to zero at infinity. This implies that

$$\lim_{x^2+z^2 \to \infty} (w_1) = 0 .$$

Another consideration, which is not so obvious, is that energy should be propagated away from the mountain. (See Appendix). This requirement is often used to make the solution unique. Alternatives to this requirement may be found in WMO Tech Note 34.

Lyra (1943) derived solutions of (6) in the simplified case of $F(z) = k_s^2$, a constant. By writing $x/M_0^{1/2}$ for x, (6) becomes

$$\nabla^2 w_1 + k_s^2 w_1 = 0 \qquad (9)$$

Since solutions are required which are centred on the mountain use plane polar co-ordinates

$$x = r \cos\theta , \quad z = r \sin\theta \qquad (10)$$

and write

$$w_1 = R(r)\,\Theta(\theta)$$

to separate (9) into

$$r^2 \frac{R''}{R} + r \frac{R'}{R} + k_s^2 r^2 = -\frac{\Theta''}{\Theta} = \Lambda \qquad (11)$$

where Λ is a constant.

The Θ - equation has solutions

$$\Theta = A \sin \Lambda^{1/2}\theta + B \cos \Lambda^{1/2}\theta \qquad (12)$$

and the R-equation is Bessel's equation (Jeffreys & Jeffreys, 1972).

Consider first the eigen-solutions of the problem. These are solutions satisfying the interior equation (9) together with the boundary conditions at infinity, but which have $w_1 = 0$ on the lower boundary. This means that any multiple of such a solution may be added to one of the full problem and the result is still a solution. Such solutions are given by

$$w_{1\nu} = a_\nu J_\nu(k_s r) \sin \nu\theta \qquad (13)$$

for ν an integer with $\nu \geq 1$. (See Jeffreys & Jeffreys, 1972, for a proof that $J_\nu(k_s r) \to 0$ as $r \to \infty$). Finding the particular solutions to which these may be added is a more difficult problem. Lyra (1943) shows how this may be done using Green's Functions. (Note: this reference is in German. Both the translation available in the Met Office library and the summary in WMO Tech Note 34 quote an incorrect result for the solution).

Lyra (1943) considered the topography to consist of step functions and used these to approximate a variety of hill shapes. Subsequently other authors, notably Queney (1947), Scorer (1949) and Zirep (1952) investigated smoother topography. Figure (2) gives some examples of Lyra's work.

The above assumed that $F(z)$ was constant throughout the atmosphere. The next approximation, before resorting to numerical methods, is to assume that the atmosphere consists of n layers in which $F(z) = l_r^2$ in the rth layer, the boundary between layer r and r + 1 lying at h_r, where $h_r > h_{r+1}$. Consider the Fourier component of the wave with wavenumber k to enable (9) to be written in the form

$$w_{1r}'' + (l_r^2 - M_0 k^2) w_r = 0 \qquad r = 1 \ldots n \qquad (14)$$

in each layer. The boundary conditions are then

w_1	continuous across	$z = h_r$
u	continuous across	$z = h_r$
$w_1 = 0$	on $z = 0$ (for an eigen-solution)	
$w_1 = u_s \frac{d\zeta_s}{dx}$	on $z = 0$ (for a particular solution)	

together with the previous constraints as $z \to \infty$. For a two layer model, such as that of Scorer (1949), this leads to

$$w_{11} = A \exp[-\mu_1 (z - h_1)] \exp ikx \qquad (15)$$

$$w_{12} = A \left(\cosh \mu_2 (z - h_1) - \frac{\mu_1}{\mu_2} \sinh \mu_2 (z - h_1)\right) \exp ikx$$

where the condition that this is an eigen-solution is only satisfied if

$$\frac{\mu_1}{\mu_2} = -\coth(h\mu_2) \qquad (16)$$

Here

$$\mu_r^2 = M_0 k^2 - l_r^2 .$$

If $M_0^2 k > l_2 > l_1$, this has no solutions, for then $\mu_1/\mu_2 < 1$. Likewise, if $l_1 > M_0^2 k > l_2$ then $(\mu_1/\mu_2)^2 < 0$ and $\coth h\mu_2 > 0$ so there are no solutions. If $M_0^2 k < l_2$ then μ_1/μ_2 is real, but μ_2 is imaginary, so that again there are no solutions. However, for $l_2 > M_0^2 k > l_1$ there is a solution if

$$(l_2^2 - M_0 k^2)^{\frac{1}{2}} \cot\left[(l_1^2 - M_0 k^2)^{1/2} h\right] = -(M_0 k^2 - l_1^2)^{1/2} \qquad (17)$$

which simplifies to

$$(2n+1+\tfrac{1}{2})\pi > (l_2^2 - l_1^2)^{1/2} h \geq (2r+\tfrac{1}{2})\pi . \qquad (18)$$

Hence, if this is satisfied for r = 0 .... N - 1 there are N lee-wave modes possible.

Foldvik (1962) has simplified Scorer's results to make them more suitable for use in forecasting. Casswell (1966) produced a series of nomograms based on this theory which enable the wavelength and vertical velocity of lee waves to be predicted.

More details of Scorers model may be found on WMO Tech Note 34.

Recent work on flow past hills has been performed by Drazin (1961), Brighton (1978), and Browning and Starr (1972). Much of the recent theoretical work has been numerical.

2. Non-linear studies of Lee Waves

The previous section discussed the linearised theory of lee waves. Although many non-linear studies have been made, most have been performed using numerical models. Long (1955) and Yih (1965) have developed analytic approaches to the problem.

Consider two-dimensional flow. The continuity equation is then

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \qquad (1)$$

where $(u', w') = \rho/\rho_0 (u, w)$ are scaled velocities and ρ_0 is a reference density. A stream function ψ' may be defined for the flow by

$$u' = \frac{\partial \psi'}{\partial z} \quad , \quad w' = -\frac{\partial \psi'}{\partial x} \qquad (2)$$

The equations of motion for a steady system are

$$\rho_0 (u' u'_x + w' u'_z) = - p_x \qquad (3)$$

$$\rho_0 (u' w'_x + w' w'_z) = - p_z - \rho g . \qquad (4)$$

Denoting the vorticity of the flow by $\eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} = \nabla^2 \psi'$ (5)

the equations of motion become

$$\rho_0 \eta' \frac{\partial \psi'}{\partial x} = \frac{\partial}{\partial x}\left[p + \rho_0 \frac{(u'^2 + w'^2)}{2} \right] \qquad (6)$$

$$\rho_0 \eta' \frac{\partial \psi'}{\partial z} = \frac{\partial}{\partial z}\left[p + \rho_0 \left(\frac{u'^2 + w'^2}{2} \right) \right] + \rho g . \qquad (7)$$

Writing these in the form of infinitesimals leads to

$$\rho_0 \eta' \, d\psi' = d\left[p + \rho_0 \left(\frac{u'^2 + w'^2}{2} \right) \right] + \rho g dz \qquad (8)$$

$$= dH - gz\, d\rho$$

where

$$H = p + \rho_0 \left(\frac{u'^2 + w'^2}{2} \right) + \rho g dz .$$

Bernouilli's theorem (Hughes and Brighton, 1967) states that H is conserved along a streamline, so that H is a function of ψ' only provided that ψ' is uni-valued in a vertical plane far upstream of the region of interest. Thus, dividing (8) by ψ'

$$\nabla^2 \psi' + \frac{gz}{\rho_0} \frac{d\rho}{d\psi'} = \frac{dH}{d\psi'} = h(\psi') , \text{ say} \qquad (9)$$

Here ρ has been assumed to be a function of ψ' alone; since the fluid is incompressible ρ is constant along a streamline.

Since h and ρ are functions of ψ' alone, they may be determined from the upstream (undisturbed) flow. Yih (1965) simplifies the theory by assuming that h and $d\rho/d\psi'$ are both linear in ψ'. This is equivalent to placing these restrictions on the basic state profile of a linear model, and allows a quadratic variation of H and ρ with height for a uniform wind. Then (9) becomes

$$\nabla^2 \psi' + z\,(C_0 + C_1 \psi') = C_2 + C_3 \psi'$$

where C_0, C_1, C_2 and C_3 are constants. The more stringent constraint that $d\rho/d\psi'$ is independent of ψ' (uniform density gradient) and that H has no linear variation in ψ, gives

$$\nabla^2 \psi' + C_0 z = C_3 \psi' . \qquad (10)$$

For motion confined to a channel of depth d

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2} \right) \psi' + C_0 d^3 \zeta = C_3 d^2 \psi' \qquad (11)$$

where

$$\xi = x/d, \quad \zeta = z/d . \quad \text{Scaling by} \quad \psi = \psi'/Ud, \quad A = \frac{C_0 d^2}{U'} = \frac{g}{\rho_0} \frac{d^2}{U'} \frac{d\rho}{d\psi'},$$

$$B = C_3 d^2$$

gives

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2} \right) \psi + A\zeta = B\psi . \qquad (12)$$

For flow from left to right and stable stratification, $A > 0$. Define

$$F^{-2} = A = -\frac{g' d}{U'^2} \quad \text{where} \quad g' = -\frac{g}{\rho_0} \frac{d\rho}{d\psi} \qquad (13)$$

and take upstream conditions

$$\psi = \psi_1 = \zeta$$

$$\rho = \rho_0 - (\rho_0 - \rho_1)\zeta$$

so that

$$B = c_3 d^2 = \frac{d^3}{\bar{\psi}} \frac{dH}{d\psi} = -F^{-2}$$

and the governing equation is

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2} \right) \psi - F^{-2} \zeta = -F^{-2} \psi. \qquad (14)$$

Long (1955) solved this problem by separation of variables. He considered a ground profile which was given by $\zeta_s = F(\xi)$, the barrier extending over $|\xi| \leq b$. In this case the upper boundary was identified with the tropopause. Long derived solutions in each of the regions

$$\xi < -b, \quad |\xi| < b, \quad \xi > b$$

by solving the problem for infinitesimal barrier height and then substituting in a finite height for the mountain. The solutions in each of the three regions were matched at the interfaces by requiring that ψ and $d\psi/d\zeta$ be continuous across the interface. The solution at the end of the sequence was not a solution to the given problem, but to a closely related one having a ground profile given by the lowest streamline in the model. Yih (1965) gives a fuller description of the technique.

Yih (1965) approaches the problem by using a series of vortices at the lower boundary to simulate topography and then solving the problem. The topography is then represented by the lowest streamline in the model. This technique suggests that the exact form of the lee wave pattern from a large obstruction depends only upon the integral properties of the barrier rather than on its exact composition.

Both Yih and Long have produced rotors in their solutions and jets occur together with regions of reverse flow for large values of F or high orography.

Appendix. Radiation boundary conditions

Numerical models of the atmosphere are designed to represent a finite region of space. Some, global models, need only to specify free conditions at the top of the atmosphere, whereas others, for example the 'Rectangle' model described by Gadd (1978), have boundaries representing computational boundaries within the flow which should not affect the flow patterns. In practice the introduction of such boundaries will cause the reflection of waves within the model unless great care is taken (Enquist and Majda, 1977).

Analytic models suffer from similar problems, even if the boundary is at infinity, as in the case of Lyra's (1943) model for lee waves. In this case the problem arises not from reflection but from the transmission of energy into the domain from infinity, whereas it would be expected that waves generated within the domain of interest should propagate energy away from the disturbance. This concept is expounded in Sommerfeld (1948).

The governing equation of Lyra's (1943) model is, from section 1,

$$\nabla^2 w_1 + k_s^2 w_1 = 0. \tag{1}$$

Look for solutions of this of the form

$$w_1 = f(z) \exp(ikx) \tag{2}$$

where the boundary conditions are not of interest here.

$$w_1 = \exp(ikx) f(z) = \begin{cases} A_{\pm} \exp i\left[kx \pm (k^2 - k_s^2)^{1/2} z\right] & \text{for } k > k_s \quad (3a) \\ \left[A \exp(k_s^2 - k^2)^{1/2} z + B \exp -(k_s^2 - k^2)^{1/2} z\right] \exp(ikx) & k < k_s \quad (3b) \end{cases}$$

The latter of these two solutions leads to exponential decay of w with z if w is to remain bounded as $z \to \infty$. The former solution is the solution with which the rest of this appendix will be concerned.

Transforming the solution (3a) to moving co-ordinates, such that x becomes $x + u_0 t$ in the new co-ordinates, gives

$$w_1 = A_{\pm} \exp i \left(kx \pm (k^2 - k_s^2)^{1/2} z + u_0 k t\right). \tag{4}$$

Write $\mu = \pm (k^2 - k_s^2)^{1/2}$

Then $$k = (\mu^2 + k_s^2)^{1/2}. \tag{5}$$

The vertical group velocity is

$$\frac{\partial u_0 k}{\partial \mu} = \frac{u_c \mu}{(\mu^2 + k_s^2)^{1/2}}. \tag{6}$$

This is upwards only if $\mu > 0$, so that the appropriate solution is

$$w_1 = A_+ \exp i\,(kx + (k^2 - k_s^2)^{1/2} z) \tag{7}$$

(in the original co-ordinates, fixed with respect to the mountain). When written in polar co-ordinates (r, θ) such that

$$x = r\cos\theta\,, \quad z = r\sin\theta$$

this becomes

$$w_1 = A_+ \exp ir(k\cos\theta + \mu\sin\theta) \tag{8}$$

so that

$$\frac{\partial w_1}{\partial r} = i\,(k\cos\theta + \mu\sin\theta)\, w_1 . \tag{9}$$

The wave crests travel parallel to the wave vector, so that for these

$$\frac{\partial w_1}{\partial r} = \frac{i\,(k^2+\mu^2)}{(k^2+\mu^2)^{1/2}}\, w_1 = i k_s w_1 . \tag{10}$$

Thus the appropriate wave form as $r \to \infty$ satisfies the differential relation (10).

Consider a wave generated by a mountain in Lyra's (1943) model. The solutions are found in section 1 to be of the form

$$w_1 \sim J_\nu(k_s r)\, f_\nu(\theta) . \tag{11}$$

Jeffreys and Jeffreys (1972) show that

$$J_\nu(k_s r) = O\left(\frac{1}{r^{1/2}}\right)$$

as $r \to \infty$ so that the appropriate form in which to apply (10) is

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial w_1}{\partial r} - i k_s w_1\right) = 0 . \tag{12}$$

Jeffreys and Jeffreys (1972) show that

$$J_\nu(k_s r) \sim \frac{1}{r^{1/2}} \exp i k_s r$$

as $r \to \infty$, so that the quoted solution also satisfies (12).

Note that in taking the solution, (11), the positive root for k_s was assumed. The boundary conditions at infinity preclude the solution with $-k_s$.

The derived solution thus ensures that the mountain acts as a source of energy, rather than a sink.

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Table of Symbols used in lecture six

1.

u_o	basic state wind
T_o	basic state temperature
θ_o	basic state potential temperature
ρ_o	basic state density
c_o	basic state speed of sound $(\gamma R T_o)^{1/2}$
u	horizontal velocity perturbation
w	vertical velocity perturbation
ρ	density perturbation
p	pressure perturbation
M_o	Mach number of basic state $\left(1 - \frac{u_o^2}{c_o^2}\right)$
S_o	$-\frac{d}{dz}(\ln \rho_o) + \frac{M_o'}{M_o}$
β_o	$\frac{d}{dz}(\ln \theta_o) + \frac{M_o'}{M_o}$
g	acceleration due to gravity
ζ_s	Surface profile
w_1	$w\left(\frac{M_o}{M_s}\frac{\rho_s}{\rho_o}\right)^{-1/2}$
$F(z)$	$\frac{\beta_o g}{u_o^2} + S_o\frac{u_o'}{u_o} - \frac{1}{4}S_o'^2 + \frac{1}{2}S_o' - \frac{u_o''}{u_o}$
k_s^2	constant value of $F(z)$
J_ν	Bessel function of first kind and ν^{th} order
ℓ_r	value of $F(z)$ in layer r
μ_r	$(M_o k^2 - \ell_r^2)^{1/2}$
k	wave number in x-direction
$(\ )_s$	surface value

2.

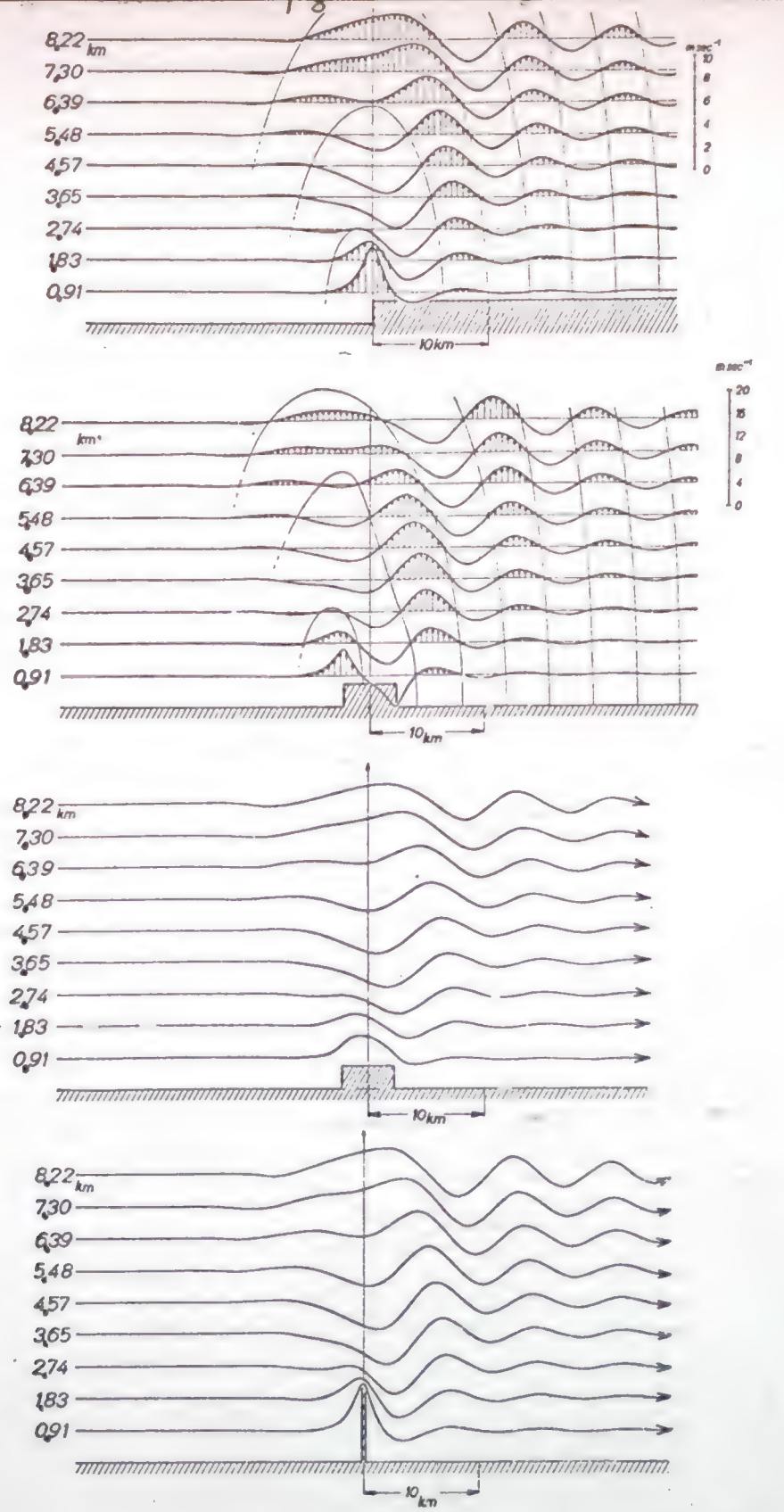
(u, w)	scaled velocities (ρ/ρ_o . velocity)
ρ_o	reference density
ρ	density
ψ'	stream function
p	pressure
η'	vorticity
g	acceleration due to gravity

H	$p + \rho_0 \frac{(u'^2 + w'^2)}{2} + \rho g z$
h	$\frac{dH}{d\psi'}$
C_0, C_1, C_2, C_3	constants giving variation of ρ and H.
d	depth of fluid
$(\xi, \zeta) = \frac{1}{d}(x, z)$	scaled co-ordinates
$\Psi = \psi'/Ud$	scaled stream function
U'	typical scaled velocity
$A = \frac{c_0 d^2}{U'}$	parameter of flow
$B = c_3 d^2$	parameter of flow
$F^{-2} = -\frac{g' d}{U'^2}$	parameter of flow
$g' = -\frac{g}{\rho_0}\frac{d\rho}{d\psi}$	parameter of flow
Ψ_1	stream function upstream
ρ_1	represents variation of density upstream
b	half-width of mountain

Appendix

As section 1 together with

$f(z)$	vertical profile of wave
$A_{\pm}$	amplitude
μ	vertical wave number



Solutions of Lyra's equation for a horizontal wind of 15 ms^{-1}

(a) Vertical velocity perturbations for a plateau.
(b) Vertical velocity perturbations for a ridge.
(c) Streamlines for a ridge.
(d) Streamlines for a vertical wall.

From Lyra (1940)

Figure 2.

LECTURE SEVEN ENERGY DISSIPATION BY LEE WAVES

1. Synoptic scale lee waves

In the previous lecture it was assumed that the mountain and perturbations produced by it were on a sufficiently small scale that the Coriolis effect could be ignored. This lecture will explore the influence of mountains with a width of order 100 km on the flow across them. Only the linear theory will be discussed.

Consider a mountain range which is uniform in a direction parallel to the y-axis and ignore the Earth's curvature. For simplicity assume f to be constant and consider an incompressible atmosphere with uniform static stability and basic state wind. Also assume that the perturbations are independent of y.

The equations governing the basic state are

$$\bar{\rho} f U + \frac{\partial \bar{p}}{\partial y} = 0 \qquad \bar{\rho} g + \frac{\partial \bar{p}}{\partial z} = 0$$

and so

$$\frac{\partial \bar{p}}{\partial z} - \varepsilon^{-1} \frac{\partial \bar{p}}{\partial y} = 0 \qquad \frac{\partial \bar{\rho}}{\partial z} - \varepsilon^{-1} \frac{\partial \bar{\rho}}{\partial y} = 0. \qquad (1)$$

Thus an isopycnic surface (one of constant density) coincides with an isobaric surface, and has slope $\tan \varepsilon$, where $\varepsilon = \frac{fU}{g}$. It can be verified that the density distribution

$$\bar{\rho} = \rho_0 \exp\left[-2s(z + \varepsilon y)\right] \qquad (2)$$

satisfies the constaints.

Consider the position of a particle at a given time, t. In the absence of a wave it would be at the position X_B under the influence of the basic state. If a wave were present its position would be X_W. The Lagrangian displacement is defined to be $X_B X_W$. The displacement will be written in the form $(\Delta x, \Delta y, \Delta z)$. The pressure experienced by the parcel at the new position is, to first order, that at the undisturbed position (δp) together with a correction due to the variation of the basic state in space, so

$$\Delta p = \delta p + \frac{\partial \bar{p}}{\partial x} \Delta x + \frac{\partial \bar{p}}{\partial y} \Delta y + \frac{\partial \bar{p}}{\partial z} \Delta z$$

where an overbar denotes a basic state quantity.

The equations of motion are then obtained by taking

$$\delta u = \frac{D}{Dt} \Delta x \qquad \text{etc}$$

and substituting into the linearised equations. The following system results

$$\frac{D}{Dt}\left[\frac{D}{Dt} \Delta x\right] - f \frac{D}{Dt} \Delta y + \frac{1}{\bar{\rho}} \frac{\partial}{\partial x} \delta p = 0 \qquad (3)$$

$$\frac{D}{Dt}\left[\frac{D}{Dt} \Delta y\right] + f \frac{D}{Dt} \Delta x + \frac{1}{\bar{\rho}}\left[\frac{\partial}{\partial y} \delta p + \frac{\partial^2 \bar{p}}{\partial y^2} \Delta y + \frac{\partial^2 \bar{p}}{\partial y \partial z} \Delta z\right] = 0 \qquad (4)$$

$$\frac{D}{Dt}\left[\frac{D}{Dt} \Delta z\right] + \frac{1}{\bar{\rho}}\left[\frac{\partial}{\partial z} \delta p + \frac{\partial}{\partial z} \frac{\partial}{\partial y} \bar{p} \Delta y + \frac{\partial^2}{\partial z^2} \bar{p} \Delta z\right] + \frac{\Delta \rho}{\bar{\rho}} g = 0 \qquad (5)$$

$$\frac{\partial}{\partial x}\left(\frac{D}{Dt} \Delta x\right) + \frac{\partial}{\partial z}\left(\frac{D}{Dt} \Delta z\right) = 0 \qquad (6)$$

$$\frac{D}{Dt}\Delta\rho = -\sigma^2 \frac{D}{Dt}\Delta z \tag{7}$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \tag{8}$$

It is possible to eliminate $\Delta\rho$ from (5) and (7).

Scale the variables according to

$$\Delta\alpha' = \Delta\alpha \exp -s(z+\varepsilon y) \qquad \text{if } \alpha \text{ is a momentum variable} \tag{9}$$

$$\Delta\beta' = \Delta\beta \exp s(z+\varepsilon y) \qquad \text{if } \beta \text{ is a thermodynamic variable} \tag{10}$$

Then, since the perturbations are to be independent of y,

$$\frac{\partial}{\partial y}\Delta\alpha' = -\varepsilon s \Delta\alpha'$$

$$\frac{\partial}{\partial y}\Delta\beta' = -\varepsilon s \Delta\beta'.$$

The equations then become

$$\frac{D}{Dt}\left(\frac{D}{Dt}\Delta x'\right) - f\frac{D}{Dt}(\Delta y') + \frac{\partial}{\partial x}\frac{\delta p'}{\rho_0} = 0 \tag{11}$$

$$\left(\frac{D}{Dt}\left(\frac{D}{Dt}\right) + \varepsilon^2\sigma^2\right)\Delta y' + f\frac{D}{Dt}\Delta x' + \varepsilon\sigma^2\Delta z' - \frac{2\varepsilon s}{\rho_0}\delta p' = 0 \tag{12}$$

$$\frac{D}{Dt}\left[\left(\frac{D^2}{Dt^2} + \sigma^2\right)\Delta z' + \varepsilon\sigma^2\Delta y' + \left(\frac{\partial}{\partial z} - s\right)\frac{\delta p'}{\rho_0}\right] = 0 \tag{13}$$

$$\frac{D}{Dt}\left[\frac{\partial \Delta x'}{\partial x} + \frac{\partial \Delta z'}{\partial y}\right] = 0 \tag{14}$$

Seek a wave of the form exp i $(kx + mz - \omega t)$, and transform to a coordinate system moving with the basic state flow, so that $\frac{D}{Dt} = \frac{\partial}{\partial t}$. This results in a dispersion relationship

$$\omega^2(k^2+m^2+s^2) - (2\varepsilon f s k)\omega - \left[(1+\varepsilon^2)\sigma^2k^2 + \varepsilon^2\sigma^2(m-is)^2 + f^2(m^2+s^2)\right] = 0 \tag{15}$$

For a wind speed of 100 ms^{-1}, with $g = 10\ ms^{-2}$ and $f \sim 10^{-4}\ s^{-1}$, the appropriate value of $\varepsilon \sim 10^{-3}$. Thus terms in ε and ε^2 may be neglected. The y-variation of the basic flow may thus be ignored for the present purposes. The dispersion relation is now

$$\omega^2(k^2+m^2+s^2) - (\sigma^2k^2 + f^2(m^2+s^2)) = 0. \tag{16}$$

These waves which are stationary with respect to the mountain correspond to $\omega = -Uk$, so that for these (16) becomes

$$m^2 = k^2\frac{(k_s^2 - k^2)}{(k^2 - k_f^2)} - s^2 \tag{17}$$

where $k_s^2 = \frac{\sigma^2}{U^2} - s^2$ and $k_f^2 = f^2/U^2$. (18)

Since s is very small, $U \ll \frac{\sigma^2}{s^2} \approx \frac{2g}{s}$, so that $k_f < k_s$.

Thus (if s^2 is neglected as small in (17)) stationary waves may occur if $m^2 > 0$, so that

$$k_f^2 < k^2 < k_s^2.$$

The vertical wavelength of the waves is not restricted to a small range since m^2 varies from 0 to infinity as k^2 ranges from k_s^2 to k_f^2.

The group velocity relative to the mountain has components $(c_{g_x}, 0, c_{g_z})$, where

$$c_{g_x} = U\left[\frac{k^2(k^2-k_f^2) + k_f^2(k_s^2+s^2-k^2)}{k^2(k_s^2+s^2-k_f^2)}\right]$$

$$c_{g_z} = U\left(\frac{m}{k}\right)\frac{(k^2-k_f^2)}{k^2(k_s^2+s^2-k_f^2)} \qquad (19)$$

Notice that the group velocity is such that the horizontal energy propagation is downstream and that the vertical component is of the same sign as Um. There is thus upward energy propagation if m is chosen to be of the same sign as U. Note that $|c_g| < U$, so that energy propagates downstream at less than the wind speed.

If $k = k_f$ the group velocity has magnitude U and the vertical wave number tends to infinity. For long waves ($k \ll k_s$) with $s^2 \ll k_s^2$, m^2

$$m^2 \approx \frac{k^2 k_s^2}{k^2-k_f^2}$$

$$\underset{\sim}{c}_g = \left(U\frac{k_f^2}{k^2}, 0, \frac{U}{k}\frac{k^2-k_f^2}{m}\right).$$

The physical significance of k_s is analagous to that of $F(z)$ in the previous lecture.

The above results may be used to derive the waves forced by a particular mountain profile. Let the height of the mountain be described by H(x), and let this be the Fourier integral of h(k), so that

$$H(x) = \mathrm{Re}\int_0^\infty h(k)e^{ikx}dk \qquad (20)$$

Assume that the solution for $\Delta z'_k$ is of the form

$$\Delta z'_k = \zeta_k(z)\exp ikz \qquad (21)$$

corresponding to each k. Then, when $\omega^2 = k^2U^2$ and $-m^2 = \frac{\partial^2}{\partial z^2}$ are substituted into (16),

$$\zeta''_k + \frac{k^2(k_s^2-k^2)}{k^2-k_f^2}\zeta_k = 0 \qquad (22)$$

If $k^2 < k_f^2$ or $k^2 > k_s^2$ the solutions of this are in terms of the hyperbolic functions, otherwise they are sinusoidal. The solution is thus

$$\Delta z'_k = A \exp(ikx \mp \mu z) \qquad \begin{cases} k^2 < k_f^2 & \text{(minus sign)} \\ k^2 > k_s^2 & \text{(plus sign)} \end{cases} \tag{23}$$

or

$$\Delta z'_k = A \exp i(kx + mz) \qquad \text{if } k_f^2 < k^2 < k_s^2$$

where

$$m = k\left[\frac{k_s^2 - k^2}{k^2 - k_f^2}\right]^{\frac{1}{2}}, \qquad \mu = k\sqrt{\frac{k^2 - k_s^2}{k^2 - k_f^2}}. \tag{24}$$

The sign of m has been chosen to give upward energy propagation. The total solution is thus given by

$$\Delta z' = Re\left\{ \int_0^{k_f} h(k) \exp(ikx - \mu z)\, dk + \int_{k_f}^{k_s} h(k) \exp i(kx + mz)\, dk + \int_{k_s}^{\infty} h(k) \exp(ikx + \mu z)\, dk \right\} \tag{25}$$

Define length-scales

$$L_s = \frac{2\pi}{k_s} \sim \frac{2\pi U}{\sigma}$$

$$L_f = \frac{2\pi}{k_f} = \frac{2\pi U}{|f|} \tag{26}$$

If $U \sim 15\ m\,s^{-1}$ these have the magnitudes $L_f \sim 1000$ km and $L_s \sim 10$ km for $f \sim 10^{-4}\ s^{-1}$. Thus $L_f / L_s \sim 100$. Queney (1973) shows how (25) may be simplified according to the relative values of b, the width of the mountain, L_f and L_s. The discussion here will be confined to $b \sim L_f$ and $b >> L_f$. In these cases the main contribution to (25) is from wave numbers near k_f. Thus the third integral in (25) may be assumed to be zero and the second to have an upper limit of infinity. In addition, k^2 may be neglected in combination with k_s^2. (25) then becomes

$$\Delta z' = Re\left[\int_0^{k_f} h(k) \exp\left\{ ikx - \frac{k_s k z}{(k_f^2 - k^2)^{1/2}} \right\} dk \right] + Re\left[\int_{k_f}^{\infty} h(k) \exp i\left\{ kx + \frac{k_s k z}{(k^2 - k_f^2)^{1/2}} \right\} dk \right]$$

and similar expressions may be found for $\Delta y'$ and $\delta p'$.

Queney (1973) discusses further approximations to this formula. Figure 1, taken from Queney (1973), shows a typical pattern for the surface streamlines and isobars. It can be seen that there is an anticyclonic region above the ridge and a lee-wave pattern downstream.

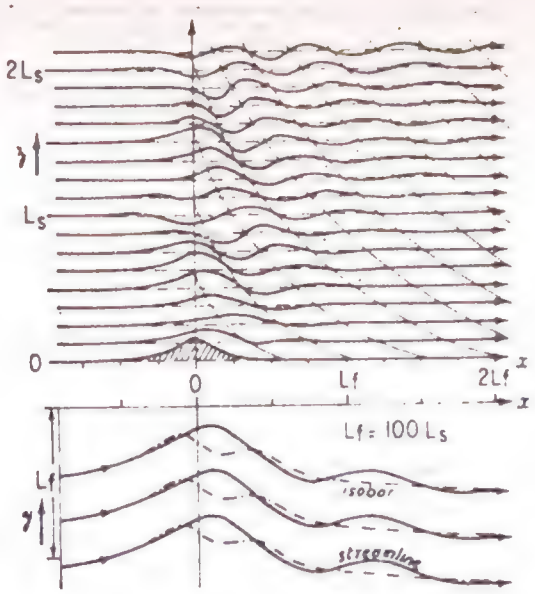


Fig. 1 - Synoptic-scale barrier perturbation
($\sigma, \bar{u}, f$ = const; northern hemisphere)

From Queney (1973)

2. Energy dissipation by synoptic-scale mountain waves

The wave motions discussed in the first part of this lecture were not confined in the vertical. Due to the simple basic state which was chosen there was no modification of the wave structure with height. In practice it would be found that the wave energy would be dissipated in the upper atmosphere by such processes as viscosity and radiative cooling. Even if these were not present the wave-structure would be come of large amplitude at great heights due to the vertical decrease in density, and would thus break up into eddies. It is thus reasonable to use the results from this simple model to investigate the energy dissipation which occurs in the atmosphere due to mountains. A further justification for this assumption is that, as will be shown, the energy dissipation only depends upon the surface perturbations.

If a frame of reference is chosen which is stationary with respect to the basic state wind the ground appears to move relative to it, and the problem remains essentially the same. Thus the problem of finding the energy dissipated by the mountain is equivalent to that of determining the work done by a mountain in moving through the atmosphere. Since there is no other energy source present in the problem this will be the energy dissipated in the dual system.

For completeness it should be stated that the linear theory assumes that the downstream flow at infinity is the same as that upstream. There are two objections to this. Firstly, if the mountain is leading to energy dissipation at a rate D, then there must be an internal source of energy which exactly balances this loss, but this is ignored in the model. Secondly, the downstream flow is unlikely to attain the same value as the upstream flow in practice, and so the assumption that the conditions are the same at both infinities is suspect. However, the energy dissipation is a second order effect, and it does not feed back upon the linearised equations. If the expansion were to be continued to higher orders of approximation, as in lecture three, it would be energy-like terms which provided the forcing for the next set of equations. The use of linear model is justifiable on these grounds.

Let P be the kinetic energy of the basic flow carried across any plane perpendicular to it in unit time. The plane is assumed to be of unit width in the y-direction. Define the relative depletion r to be W/P, where W is the work done by the mountain on the flow. P is given by

$$P = \frac{U}{2}\int_0^\infty \bar{\rho}\, U^2 dz = \frac{\rho_0 U^3}{4s} \qquad (1)$$

The work done on the fluid by the mountain in the absence of viscosity is solely that due to the pressure. Thus, if the surface is described by $z = H(x)$, where $H(\pm\infty) = 0$,

$$W = U\int_{-\infty}^{\infty} (\bar{p}_0 + \Delta p_0)\, H'(x)\,dx = U\int_{-\infty}^{\infty} \Delta p_0\, H'(x)\,dx \qquad (2)$$

where the zero subscript is used to denote a surface value, Δp is the Lagrangian perturbation, so that

$$\Delta p = \delta p + \frac{\partial \bar{p}}{\partial y}\Delta y + \frac{\partial \bar{p}}{\partial z}\Delta z = \delta p - \bar{\rho} f U \Delta y - \bar{\rho} g \Delta z \qquad (3)$$

Using $\rho \frac{\partial u}{\partial t} - f\rho v + \frac{\partial p}{\partial x} = 0$ together with the incompressibility condition $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ it may be shown that

$$\delta p = \bar{\rho} U^2 \frac{\partial \Delta z}{\partial z} + f\bar{\rho} U \Delta y$$

and hence that

$$\Delta p = \bar{\rho} U^2 \left(\frac{\partial}{\partial z} - \frac{g}{U^2}\right)\Delta z - \bar{\rho} U^2 \left[\frac{\partial}{\partial z} - \frac{g}{U^2} + s\right]\Delta z'.$$

Substituting surface values into this gives

$$\Delta p_0 = \rho_0 U^2 \left(\frac{\partial}{\partial z}\Delta z'\right)_0 - \rho_0 U^2 \left(g/U^2 - s\right) H(x)$$

as $\Delta z'_0 = H(x)$. Hence

$$r = \frac{W}{\bar{p}} = 4s\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial z}\Delta z'\right)_0 H'(x)\,dx - 4s\left(\frac{g}{U^2} - s\right)\int_{-\infty}^{\infty} H(x)\,H'(x)\,dx .$$

Thus, using $H(\pm\infty) = 0$ and integrating by parts

$$r = 4s\int_{-\infty}^{\infty} \left(-\frac{\partial^2}{\partial x \partial z}\Delta z'\right)_0 H(x)\,dx . \qquad (4)$$

Using $\Delta z'$ from the previous section, the first and third of the corresponding integrals for r may be shown to be zero (Queney, 1973). Thus the only contribution comes from the second integral. Thus

$$r = 4\pi s \int_{k_f}^{k_s} k m\, |h(k)|^2\, dk$$

$$= 4\pi s \int_{k_f}^{k_s'} k\left(k^2 + \frac{s^2 k_f^2}{k_s'^2}\right)^{\frac{1}{2}} \left(\frac{k_s'^2 - k^2}{k^2 - k_f^2}\right)^{\frac{1}{2}} |h(k)|^2\, dk \qquad (5)$$

where $k_s' = \left[\frac{1}{2}k_s^2 + \left(k_s^4 + 4s^2 k_f^2\right)^{\frac{1}{2}}\right]^{1/2}$.

This has a pole at $k = k_f$ which may be removed by writing

$$\theta = (k^2 - k_f^2)^{1/2} / k_s'$$

and using this as the variable of integration. The expression for τ then becomes

$$\tau = 4\pi s k_s'^2 \int_0^1 \left[k_s'^2 \theta^2 + k_f^2 \left(1 + \frac{s^2}{k_s'^2}\right) \right]^{\frac{1}{2}} (1-\theta^2)^{\frac{1}{2}} \left| h\left[\left(k_s'^2 \theta^2 + k_f^2 \right)^{1/2} \right] \right|^2 d\theta$$

where k_f^2 has been neglected when in combination with $k_s'^2$.

For a mountain profile of the form

$$H(x) = h\, e^{-8x^2/b^2}$$

$$h(k) = \frac{hb}{\sqrt{8\pi}} \exp -(b^2k^2/32) \qquad (6)$$

where h represents the height and b the width, it can be shown that

$$\tau = \frac{s h^2}{b} R(\lambda)$$

for $U \ll \frac{\sigma}{s}$, $k_s \gg \frac{\sigma}{U}$, $k_f = \frac{f}{U}$. Here

$$\lambda = \frac{\sigma}{4}\frac{b}{U} = \frac{2\pi b}{2\gamma L_f}, \quad L_f = \frac{2\pi U}{f}$$

$$R(\lambda) = 32\lambda^3 \int_0^1 \left[(\theta^2+\gamma^2)(1-\theta^2)\right]^{\frac{1}{2}} \exp -\lambda^2(\theta^2+\gamma^2)\, d\theta$$

and $\gamma = b/\sigma$

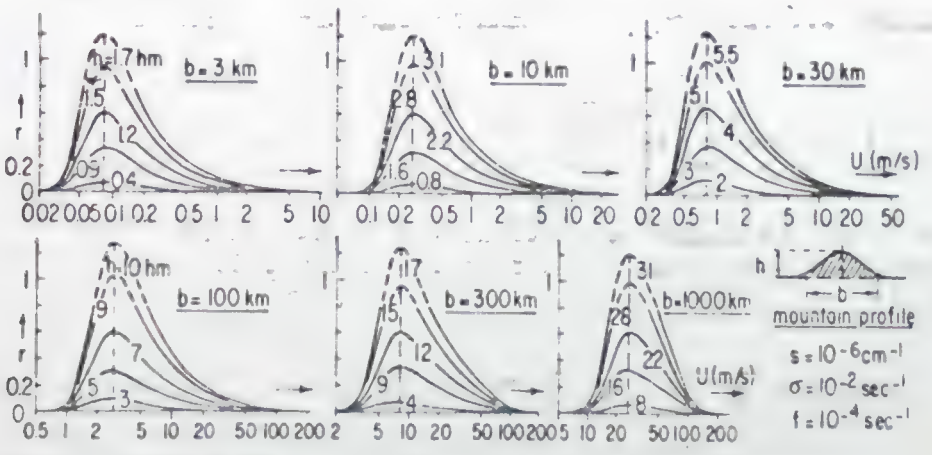


Fig. 2 - Depletion of the mechanical energy of a steady uniform wind crossing a mountain range (r = dissipated energy per unit wind energy) From Queney (1973)

Figure 2 shows the variation of τ with U for different values of h and b. It can be seen that, for fixed σ, f and s, that r is a function of U/b. It should be remembered that the theory is that for linearised equations, so that the results given for large values of r are likely to be in error (and it is not physically meaningful to have $r > 1$). With this proviso it is possible to deduce that, as the wind speed increases from zero, r increases rapidly until it attains a maximum. Further increasing the wind speed decreases r.

Queney (1973) shows that since γ is small, $R(\lambda)$ may be approximated in terms of known functions. Using such approximations he shows that the maximum value of r corresponds to mountains of the scale considered in the previous section, and not those of smaller size. This can be attributed to the greater upward energy flux of the former. Values of r for different values of U, b and h are shown in Table 1.

b(km)	100				1000			
h(m)	700				2200			
U(ms^{-1})	1	2.7	10	50	10	27	76	108
P(kJ)	3	60	3063	380×10^6	3063	60×10^6	1×10^9	4×10^9
r(%)	<2	62	20	4	< 2	62	20	4

Values of the relative depletion of the energy of the basic flow by mountain waves. Based on Queney (1973).

$\rho_0 = 1.225$ kg m^3, $\gamma = 10^{-2}$, $s = 10^{-4} m^{-1}$, $\sigma = 10^{-2} s^{-1}$ and $f = 10^{-4} s^{-1}$

Table 1

The frictional effects of topography are thus large, so that the correct representation of mountains on numerical models is important for the energy balance.

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Table of symbols used in lecture seven

A	amplitude of wave.
b	width of mountain.
$c_0 = \frac{\sigma^2}{s^2}$	
$\underline{c}_g = (c_{gx}, c_{gz})$	group velocity
$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$	time derivative following flow.
f	Coriolis parameter.
g	acceleration due to gravity.
$H(x)$	ground shape
h	mountain amplitude
$h(k)$	inverse Fourier transform of $H(x)$
k	horizontal wave number
$k_s = \left(\frac{\sigma^2}{U^2} - s^2\right)^{1/2}$.	reference wave-numbers.
$k_f = \frac{\lvert f \rvert}{U}$	reference wave-numbers.
L_f	length scales associated with k_s and k_f.
L_s	length scales associated with k_s and k_f.
m	vertical wave number
P	kinetic energy flux of basic flow.
$\bar{P}$	basic state pressure
Δp	Lagrangian pressure perturbation.
δp	Eulerian pressure perturbation.
$r = w/P$	Relative depletion.
$s = \frac{1}{2\bar{\rho}}\frac{d\bar{\rho}}{dz}$	Scale-height for density.
U	velocity of basic state flow.
δu	Eulerian velocity perturbation.
$\Delta x, \Delta y, \Delta z$	Lagrangian displacements.
$\Delta\alpha, \Delta\beta$	dummy variables.
$\bar{\rho}$	basic state density
ρ_0	surface density of basic state
$\varepsilon = \frac{fU}{g}$	slope of basic state isopycnics

$\sigma^2 = \frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz} = 2gs$	square of Brunt-Väisälä frequency.
ω	angular frequency
μ	decay with height of scaled wave.
$\zeta_k(z)$	vertical structure of wave with wave-number k.
$(\ )_o$	surface variable
$(\ )'$	scaled variable

LECTURE EIGHT THE THEORY OF RAY TRACING

1. Ray tracing in a fluid at rest

Consider a region in which waves, for example sound waves, may be written in the form

$$\phi = \Phi(\underset{\sim}{x},t)\ \exp i\{\alpha(\underset{\sim}{x},t)\} \tag{1}$$

and define

$$k_i = -\frac{\partial \alpha}{\partial x_i}\ , \quad \omega = \frac{\partial \alpha}{\partial t} \tag{2}$$

Provided that Φ only varies slowly with position the wave will appear to be locally sinusoidal with local wavenumber $\underset{\sim}{k}$ and angular frequency ω. The dispersion relation is of the form

$$\omega = \omega(\underset{\sim}{k},\underset{\sim}{x}) \tag{3}$$

and a 'group velocity', $\underset{\sim}{U}$, may be defined by

$$\underset{\sim}{U} = \frac{\partial \omega}{\partial \underset{\sim}{k}} \tag{4}$$

combining (2) and (3) gives

$$\frac{\partial \alpha}{\partial t} = \omega\left(-\frac{\partial \alpha}{\partial x_1}, -\frac{\partial \alpha}{\partial x_2}, -\frac{\partial \alpha}{\partial x_3}\ ;\ x_1, x_2, x_3\right)$$

and so

$$\frac{\partial^2 \alpha}{\partial t \partial x_i} = -\frac{\partial \omega}{\partial k_j}\frac{\partial^2 \alpha}{\partial x_j \partial x_i} + \frac{\partial \omega}{\partial x_i}$$

where the summation convention has been used.

Thus

$$\frac{\partial k_i}{\partial t} + U_j \frac{\partial k_i}{\partial x_j} + \frac{\partial \omega}{\partial x_i} = 0$$

Hence

$$\frac{D}{Dt} x_j = \frac{\partial \omega}{\partial k_j} \tag{5}$$

$$\frac{D}{Dt} k_j = -\frac{\partial \omega}{\partial x_j} \tag{6}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x_j}$.

These equations are similar to Hamilton's equations for the motion of a system of particles. Define a ray to be the curve along which energy is transmitted between two points by the wave. Along a ray

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial x_i}\frac{Dx_i}{Dt} + \frac{\partial \omega}{\partial k_i}\frac{Dk_i}{Dt} = \frac{\partial \omega}{\partial x_i}\frac{\partial \omega}{\partial k_i} - \frac{\partial \omega}{\partial k_i}\frac{\partial \omega}{\partial x_i} = 0,$$

and so the frequency is conserved along a ray. Note that the wave-number is not conserved if $\frac{\partial \omega}{\partial x_i}$ varies with position.

2. Ray tracing in a moving fluid

The above discussion was for a fluid at rest. In this section the results will be generalised to the case of an atmosphere with a mean wind $\underset{\sim}{V}(\underset{\sim}{x})$ which is assumed to vary slowly on scales comparable to the wavelength. For each point it is possible to find a co-ordinate transformation which locally reduces the mean flow to rest. This Gallilean transform is in general different for each point in the fluid.

Define the relative frequency, ω_r, to be a function of $\underset{\sim}{x}$ and $\underline{k}$ only which represents the frequency of the wave relative to the local co-ordinate system. Once again, assume that the variations are small on the scale of wavelengths.

Write

$$\phi = \Phi \exp i\{\alpha(\underset{\sim}{x}, t)\}. \tag{1}$$

α is known as the 'eikonal'. As before define the wave number by

$$\frac{\partial \alpha}{\partial x_i} = -k_i \tag{2}$$

and the absolute frequency by

$$\frac{\partial \alpha}{\partial t} = \omega. \tag{3}$$

The group velocity relative to the local frame of reference is $\frac{\partial \omega_r}{\partial k_i} = U_i$. ω_r is the rate at which the phase changes for a point moving with velocity $\underset{\sim}{V}$, so that

$$\omega_r = \frac{\partial \alpha}{\partial t} + V_i \frac{\partial \alpha}{\partial x_i} = \omega - V_i k_i. \tag{4}$$

This is the Doppler effect, and waves emitted by a moving source also obey this relationship.

The governing equations for a ray may now be written as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (U_i + V_i)\frac{\partial}{\partial x_i}$$

$$\frac{D}{Dt} x_i = V_i + \frac{\partial \omega_r}{\partial k_i} \tag{5}$$

$$\frac{D}{Dt} k_i = -k_j \frac{\partial V_j}{\partial x_i} - \frac{\partial \omega_r}{\partial x_i} \tag{6}$$

It can easily be seen that the absolute frequency, ω, is conserved along a ray. The relative frequency, ω_r, is not constant.

3. Example

In order to illustrate the techniques of ray tracing, consider the paths taken by sound waves. The first case which will be investigated is that of a stratified atmosphere, the basic state of which is uniform in the horizontal.

The velocity of the basic state is

$$\underset{\sim}{V} = (V(z), 0, 0) \tag{1}$$

after a suitable choice of axes. The speed of sound is given by

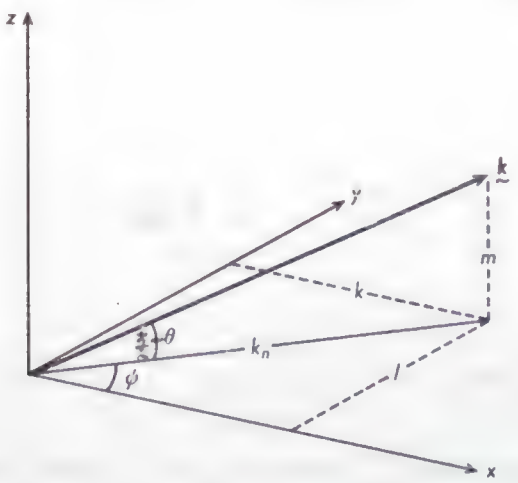
$$c_0(z) = \sqrt{\gamma R T_0(z)} \tag{2}$$

The relative frequency is thus $\omega_r = \omega - kV(z)$ the wave vector being (k, l, m). Thus the ray paths are given by

$$\frac{Dx}{Dt} = V(z) + \frac{\partial \omega_r}{\partial k} \qquad \frac{Dk}{Dt} = -k\frac{\partial V}{\partial x} = 0$$

$$\frac{Dy}{Dt} = \frac{\partial \omega_r}{\partial l} \qquad \frac{Dl}{Dt} = -k\frac{\partial V}{\partial y} = 0$$

$$\frac{Dz}{Dt} = \frac{\partial \omega_r}{\partial m} \qquad \frac{Dm}{Dt} = -k\frac{\partial V}{\partial z} - \frac{\partial \omega}{\partial z}$$



Definition sketch for ray tracing example.

Figure 1.

If k_n, θ and Ψ are as defined in figure 1, then

$$\omega_r = c_0(z)\left(k^2 + l^2 + m^2\right)^{\frac{1}{2}} = k_n\, c_0(z) \operatorname{cosec} \theta, \tag{3}$$

so that $\omega = k_n \left[c_0(z) \operatorname{cosec} \theta + V(z) \cos \Psi \right]$

which gives

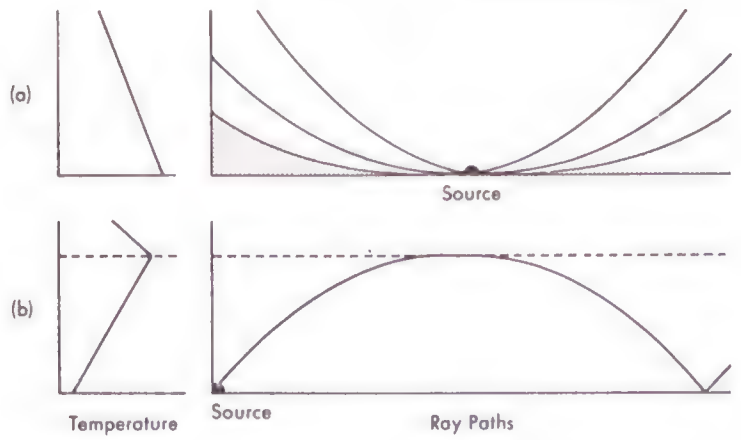
$$\sin \theta = \frac{c_0(z)}{(\omega/k_n) - V(z)\cos\Psi}, \tag{4}$$

In this form the refraction of the rays due to the temperature variations is represented in the numerator, and that due to the velocity is found in the denominator. The rays may be found from

$$\frac{dx}{dz} = V(z)\left[c_0(z)\right]^{-1} \sec\theta + \cos\psi \tan\theta$$

and

$$\frac{dy}{dz} = \sin\psi \tan\theta .$$



(a) $c_0(z)$ decreases with height. Thus $\sin\theta$ decreases with height. The shaded area, called the "zone of silence", lies below the ray which is initially tangential to the ground.

(b) $c_0(z)$ initially increases with height. Thus for small θ the rays may be refracted back to the ground.

(Schematic, after Lighthill (1978))

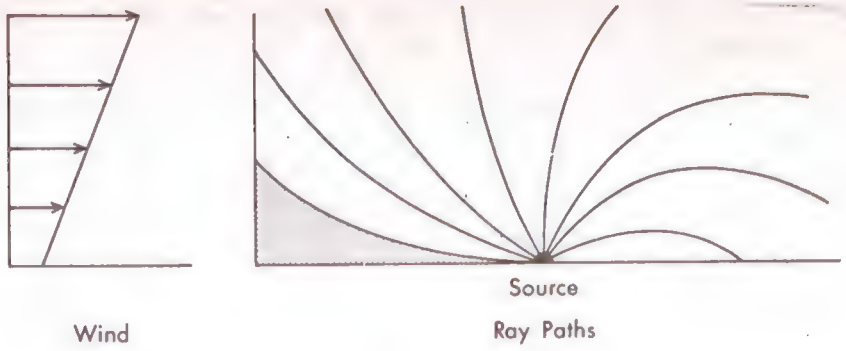
Figure 2.

Figure 2 shows ray paths for different temperature structures of the atmosphere. The technique described here is strictly not applicable near the zone of silence, although Lighthill (1978) shows that the WJKB approximation allows this deficiency to be overcome. Although the zone of silence is not completely silent, the intensity of the sound within it is markedly less than that outside. The differences in ray paths between the two cases illustrated would have marked consequences for an observer near an explosion. The explosion taking place in an atmosphere with a constant lapse rate would be almost inaudible to an observer in the zone of silence, whereas an observer in the same position under a temperature inversion would hear sound waves of high intensity.

The effects of a vertical wind shear are similar to those of a temperature gradient, but have an asymmetry introduced by the direction of the wind. For convenience consider only those waves which are propagating in the plane of the wind, so that $\cos\psi = 1$. Then

$$\sin\theta = \frac{c_0(z)}{\omega k_h^{-1} - V(z)}$$

where $c_0(z)$ is now assumed to be independent of height. For waves which are propagating downwind under conditions in which $V(z)$ increases with height, $\sin\theta$ decreases, thus trapping the waves. Upwind propagating waves under the same conditions follow ray paths which curve upwards. The roles of upwind and downwind propagation are interchanged in an atmosphere with wind decreasing with height. These effects are illustrated in figure 3.



The propogation of wave energy in an atmosphere with constant velocity of sound and a wind which increases with height. The shaded area represents the zone of silence. (Schematic)

Figure 3.

4. Wave action

Lighthill (1978) shows how the energy of sound and internal gravity waves vary along rays. He states that the wave energy per unit volume is given by

$$W = \underbrace{\tfrac{1}{2}\left[\rho_0(z)\right]\underset{\sim}{u}\cdot\underset{\sim}{u}}_{\text{kinetic energy}} + \underbrace{\tfrac{1}{2}\left[c_0(z)\right]^{-2}\left[\rho_0(z)\right]^{-1}p_e^2}_{\text{acoustic wave energy}} + \underbrace{\tfrac{1}{2}\left[\rho_0(z)\right]\left[N(z)\right]^2\}^2}_{\text{internal wave energy}}$$

the symbols taking the meanings given in the table of symbols. By partially differentiating this with respect to time, Lighthill (ibid) shows that

$$\frac{\partial W}{\partial t} = -\underset{\sim}{\nabla}\cdot\underset{\sim}{I} \qquad \text{where } \underset{\sim}{I} = p_e\underset{\sim}{u} \qquad (1)$$

and also that $\underset{\sim}{I} = W\underset{\sim}{U}$, where $\underset{\sim}{U}$ is the group velocity.

Thus

$$\frac{\partial W}{\partial t} = -\underset{\sim}{\nabla}\cdot(W\underset{\sim}{U})$$

For waves of fixed ω , including wave components deduced by Fourier analysis of the total waveform, the wave pattern appears fixed in time. That is, the wave energy density at a point remains constant. Thus

$$\underset{\sim}{\nabla}\cdot(W\underset{\sim}{U}) = 0,$$

and hence, by Gauss' theorem, if A is the cross-sectional area of a 'ray tube' with walls made up of rays

$$WUA = \text{constant}$$

along the tube.

For a dispersive system

$$\frac{Dw}{Dt} = - w \underset{\sim}{\nabla}.\underset{\sim}{U}$$

where $\frac{D}{Dt}$ is the derivative following a ray.

For the case of ray tracing in a wind the situation is more complicated. It can be shown that

$$\frac{\partial W_r}{\partial t} = - \frac{\partial W_r U_i}{\partial x_i} - \rho_0 \overline{u_i u_j} \frac{\partial V_j}{\partial x_i}$$

and hence

$$\frac{\partial W_r}{\partial t} = - \frac{\partial W_r U_i}{\partial x_i} - \omega_r^{-1} k_j I_j \frac{\partial V_j}{\partial x_i}$$

and thus

$$\frac{D}{Dt} (W_r \omega_r) = - (W_r \omega_r^{-1}) \frac{\partial U_i}{\partial x_i} .$$

This states that the wave action $W_r \omega_r^{-1}$ is unchanged along rays. For a detailed derivation of this result see Lighthill (1978). This conservation implies that when rays move into regions of greater ω_r the wave energy increases, and vice versa.

5. Method of characteristics

One technique for the solution of partial differential equations is the method of characteristics. For hyperbolic equations, such as the wave equation, the characteristics are curves along which energy may be propagated. Regarding wave energy to be the information in the solution of the equation the immediate conclusion may be drawn that the rays of ray tracing techniques may be identified with the characteristics of the governing equation. (Landau and Lifshitz, 1959). Thus the theory developed above for the tracing of sound waves may also be used for tracing shock waves in the atmosphere. It is beyond the scope of this lecture to give a detailed discussion of supersonic flow, but further information may be found in Landau and Lifschitz (1953), Lighthill (1978) and Hughes and Brighton (1967).

The correspondence between characteristics and rays allows the paths of sonic shocks to be modelled using ray tracing. An example of this application of the theory may be found in Nicholls and James (1971) where the techniques of ray tracing are used to determine the positions at which the sonic bang from a supersonic aircraft would be audible.

6. Critical layers

This section will discuss the phenomenon of a critical level. The theory will be presented for internal gravity waves, but it is applicable to all types of wave motion. A critical level occurs in a fluid when the fluid velocity and the phase velocity of the wave are identical. This frequently leads to singularities in the governing equations, and as is shown later these correspond to a physical source or sink of wave energy. The concept of a critical level has a venerable ancestry, being first commented on by Kelvin in 1880.

This discussion follows Bretherton (1966).

Consider the linearised equations of motion in height co-ordinates. Defining the Brunt-Väisälä frequency

$$N^2 = - \frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}$$

as in the previous lecture and

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}$$

and linearising the equations about the basic state $(U(z), V(z), 0)$ whilst making the Boussinesq assumption (Hide, 1978) gives

$$\frac{D_0}{Dt} u' + U_z w' + \frac{1}{\bar{\rho}} p'_x = 0 \qquad (1)$$

$$\frac{D_0}{Dt} v' + V_z w' + \frac{1}{\bar{\rho}} p'_y = 0 \qquad (2)$$

$$\frac{D_0}{Dt} w' + \sigma' + \frac{1}{\bar{\rho}} p'_z = 0 \qquad (3)$$

$$u'_x + v'_y + w'_z = 0 \qquad (4)$$

$$\frac{D_0}{Dt} \sigma' - N^2 w' = 0 \qquad (5)$$

where $\sigma' = \frac{g\rho'}{\bar{\rho}}$ is the buoyancy force. A prime denotes a perturbation quantity.

Taking the divergence of $\nabla p'$ as given by (1) - (3) gives

$$\frac{1}{\bar{\rho}} \nabla^2 p' + 2U_z w'_x + 2V_z w'_y + \sigma'_z = 0 \qquad (6)$$

and $\frac{D_0}{Dt} \nabla^2 (4)$ when p' and σ' have been substituted for yields

$$\frac{D_0^2}{Dt^2} \nabla^2 w' - \frac{D_0}{Dt}\left[U_{zz} w'_x + V_{zz} w'_y\right] + N^2\left[w'_{xx} + w'_{yy}\right] = 0. \qquad (7)$$

Define a new time variable $\tau = \varepsilon t$

and write

$$w' = Re\left[(w + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots) \exp(i\varepsilon^{-1}\phi)\right] \qquad (8)$$

as $\varepsilon \to 0$. Then $w(x, y, z, t)$ is the amplitude of a locally sinusoidal wave with phase $\varepsilon^{-1} \phi(x, y, z, t)$. εw_1, $\varepsilon^2 w_2$ are higher order corrections to the amplitude, but they will not be used here. As in the previous section, define

$$\omega = \varepsilon^{-1} \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \tau} \tag{9}$$

$$\underset{\sim}{k} = (k, l, m), \quad k = \varepsilon^{-1} \frac{\partial \phi}{\partial x}, \quad l = \varepsilon^{-1} \frac{\partial \phi}{\partial y}, \quad m = \varepsilon^{-1} \frac{\partial \phi}{\partial z}. \tag{10}$$

Define new basic state velocities

$$U^* = \varepsilon^{-1} U, \quad V^* = \varepsilon^{-1} V$$

and define a local Richardson number

$$R = \frac{N^2}{U_z^2 + V_z^2}$$

which tends to infinity as $\varepsilon \to 0$ in order that U and V do not tend to zero.

Then, equating terms in ε^{-2}

$$\left[(\phi_\tau + U^* \phi_x + V^* \phi_y)^2 (\phi_x^2 + \phi_y^2 + \phi_z^2) - N^2 (\phi_x^2 + \phi_y^2) \right] w \exp(i \varepsilon^{-1} \phi) = 0$$

The terms in the square brackets must sum to zero for non-trivial w, and so

$$\omega = kU + lV \pm \left[\frac{k^2 + l^2}{k^2 + l^2 + m^2} \right]^{\frac{1}{2}} N. \tag{11}$$

This is a local disperson relationship. It may be written in the form

$$\omega = \Omega(k, l, m, z)$$

where $\omega = \omega(x, y, z, t)$ and $\underset{\sim}{k} = \underset{\sim}{k}(x, y, z, t)$. Define the group velocity to be

$$\underset{\sim}{u}_g = (u_g, v_g, w_g) = \left(\frac{\partial \Omega}{\partial k}, \frac{\partial \Omega}{\partial l}, \frac{\partial \Omega}{\partial m} \right)$$

and an associated derivative

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} + w_g \frac{\partial}{\partial z}.$$

Thus

$$k_t = \varepsilon^{-1} \phi_{xt} = -\omega_x = -\frac{\partial \Omega}{\partial k} k_x - \frac{\partial \Omega}{\partial l} l_x - \frac{\partial \Omega}{\partial m} m_x$$

$$= -u_g k_x - v_g l_x - w_g m_x$$

$$= -u_g \phi_{xx} - v_g \phi_{xy} - w_g \phi_{xz}$$

$$= -u_g k_x - v_g k_y - w_g k_z$$

and similarly for l and m. Thus $\frac{D_g k}{Dt} = 0$.

Thus an observer travelling with the group velocity will always see waves with the same wave number. Similarly

$$\frac{D_g}{Dt}\omega = 0 \quad , \quad \frac{D_g l}{Dt} = 0 \quad , \quad \frac{D_g}{Dt} m = -\frac{\partial \Omega}{\partial z} . \tag{12}$$

Substituting from (11) gives

$$\underset{\sim}{u}_g = \left(U \pm \frac{N k m^2}{(k^2+l^2)^{\frac{1}{2}}(k^2+l^2+m^2)^{3/2}} \; , \; V \pm \frac{N l m^2}{(k^2+l^2)^{\frac{1}{2}}(k^2+l^2+m^2)^{3/2}} \; , \; \pm \frac{N (k^2+l^2) m}{(k^2+l^2)^{\frac{1}{2}}(k^2+l^2+m^2)^{3/2}} \right) \tag{13}$$

which is just the group velocity for gravity waves in a stationary atmosphere, superposed on the basic state wind. (Lighthill, 1978).

Taking the vertical component of group velocity and using (11) to eliminate m gives

$$w_g = \pm \left\{ 1 - \frac{(\omega - kU - lV)^2}{N^2} \right\}^{\frac{1}{2}} \frac{(\omega - kU - lV)^2}{N (k^2+l^2)^{1/2}} \; . \tag{14}$$

Near the level z_c at which $\omega = kU + lV$

$$w_g \underset{\sim}{\propto} \pm \frac{(kU_z + lV_z)^2}{N} \cdot \frac{(z-z_c)^2}{(k^2+l^2)^{1/2}} .$$

Thus, for an upward travelling group to travel from z_1 to z_2 takes a time

$$t_2 - t_1 \underset{\sim}{\propto} \frac{N (k^2+l^2)^{\frac{1}{2}}}{(kU_z + lV_z)^2} \left[\frac{1}{z_2 - z_c} - \frac{1}{z_1 - z_c} \right] \tag{15}$$

which is arbitrarity large as $z_2 \to z_c$. Thus a <u>wave group will never reach the critical level</u>. The vertical wavelength $2\pi/m$ decreases as the critical level is approached, but the horizontal wavelength is unaltered. The horizontal group velocity relative to the mean flow thus decreases to zero, and the absolute group velocity normal to the wavefront becomes independent of height:

$$k u_g + l v_g \underset{\sim}{\propto} kU_c + lV_c + O(z - z_c)^2$$

whereas that parallel to it is just

$$l u_g - k v_g = lU - kV$$

at every height z.

Having decided that the wave energy never reaches a critical level it is now necessary to determine what happens to the wave energy as the wave approaches the critical level. Consider a wave packet such that k, l, m and ω are independent of position but for which the amplitude is spatially varying. Then z_c is constant for each part of the group. If at t_1 a portion of the group occupied a region δz_1, and its boundaries move with their respective group velocities, then at t_2 it occupies δz_2 where, from (15)

$$\frac{\delta z_2}{(z_2 - z_c)^2} = \frac{\delta z_1}{(z_1 - z_c)^2} .$$

Bretherton (1966) states that the ratio of wave energy densities may be found from

$$\frac{\bar{E}_2 \, \delta z_2}{(kU_2 + lV_2)(z_2 - z_c)} = \frac{\bar{E}_1 \, \delta z_1}{(kU_2 + lV_2)(z_1 - z_c)}$$

so that

$$\frac{\bar{E}_2}{\bar{E}_1} = \frac{z_1 - z_c}{z_2 - z_c}$$

and the wave energy density increases as the critical level is approached. However, it can be shown (Bretherton, ibid) that the wave energy is proportional to $\omega - Uk - Vl$, which tends to zero as the critical level is approached.

The interpretation usually given to these results is that, for internal gravity waves, a critical level acts as an absorber of wave energy, and there is no reflection.

It is interesting to contrast this with another form of critical level which occurs when

$$\omega - kU - lV = \pm N .$$

There m becomes very small, and becomes imaginary for z exceeding some maximum or minimum value z_m. In this neighbourhood

$$m^2 \approx \pm (k^2 + l^2) \frac{(kU_2 + lV_2)}{N} (z - z_m)$$

where the sign is positive if, z_m is a minimum, and negative if it is a maximum. Total internal reflection occurs at z_m, for ω cannot exceed N for an internal gravity wave. The time taken for the group to reach z_m and be reflected is finite, because

$$w_g \propto |z - z_m|^{-1/2} ,$$

so that the time taken is proportional to $|z - z_m|^{1/2}$.

Discussions of critical levels may be found in Gossard and Hocke (1975), Mysak and Le Blond (1978), and Lighthill (1978).

7. Critical lines

Béland (1978) has integrated the barotropic equations numerically in the presence of a critical line for which $U=0$. His results are illustrated in figure 4. It is possible to see the 'cats eye' pattern which is common to many critical layer phenomena. For the case of Rossby waves it is as yet undecided whether a critical line acts as an absorber or reflector. Beland's results indicate that, given sufficient time, (16 days for a zonal wind, varying between $\pm$ 50 ms^{-1} and 'non-dimensional amplitude' 0.2). The critical line will act as a reflector in the presence of viscosity. However, Hoskins (personal communication) suggests that the atmosphere does not attain this state due to the action of transients and that in practice the critical line acts as an absorber.

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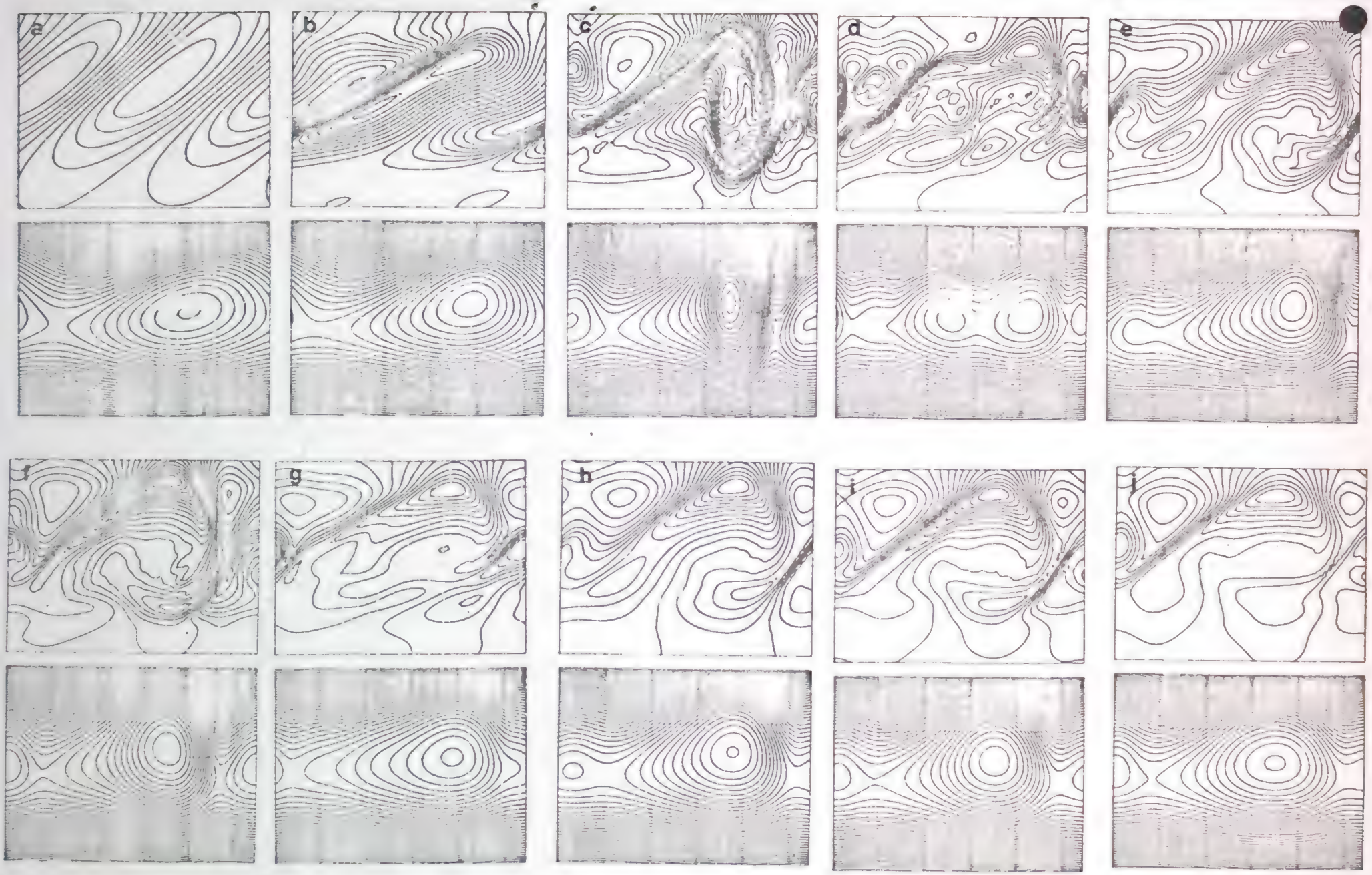


Time evolution of a Rossby wave in the presence of a critical line. Stream function (bottom) and vorticity (top) for large amplitude perturbations at non-dimensional times (a)8,(b) 17,(c) 25,(d) 34 (e) 42,(f) 51,(g) 60,(h) 69,(i) 77,(j) 86.

From Béland (1978)

Figure 6.

Table of symbols for lecture eight

1-5

A	cross-sectional area of ray tube
$c_o(z) = (\gamma R T_o(z))^{1/2}$	basic state speed of sound
g	acceleration due to gravity
$\underset{\sim}{I} = p_e \underset{\sim}{u}$	wave intensity
$k_i = -\frac{\partial \alpha}{\partial x_i}$	ith component of wave vector
k_h	horizontal projection of wave vector
$N(z) = -\left(\frac{g}{\rho_o}\frac{d\rho_o}{dz}\right)^{\frac{1}{2}}$	Brunt-Väisälä frequency
p_e	perturbation pressure
R	gas constant
$T_o(z)$	basic state temperature
$\underset{\sim}{u}$	perturbation velocity
$\underset{\sim}{U} = \frac{\partial \omega}{\partial \underset{\sim}{k}}$	group velocity
$\underset{\sim}{V}$	basic state wind
$V(z)$	basic state wind
W	energy density of wave
W_r	relative energy density of wave
$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x_j}$	derivative along a ray
$\alpha(\underset{\sim}{x}, t)$	eikonal
γ	ratio of specific heats
ϕ	potential function
$\Phi(\underset{\sim}{x}, t)$	amplitude function
Ψ	angle of wave vector to horizontal wind
θ	angle of wave vector to vertical
$\rho_o(z)$	basic state density
ρ	perturbation density
$\omega = \frac{\partial \alpha}{\partial t} = \omega(\underset{\sim}{k}, \underset{\sim}{x})$	angular frequency
ω_r	relative frequency
ζ	height perturbation
$\overline{(\ )}$	mean

$\bar{E}_i$	energy density at t_i
g	acceleration due to gravity
$\underset{\sim}{k} = (k,l,m) = \varepsilon^{-1} \frac{\partial \phi}{\partial \underset{\sim}{x}}$	wave vector
$N = -\left(\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz} \right)^{1/2}$	Brunt-Väisälä frequency
p'	pressure perturbation
$R = \frac{N^2}{U_z^2 + V_z^2}$	local Richardson number
t_i	time
$(U,V,0)$	basic state wind
$(U^*, V^*) = \varepsilon^{-1} (U,V)$	scaled wind
$\underset{\sim}{u}_g = (u_g, v_g, w_g)$	group velocity
(u', v', w')	velocity perturbation
$w' = Re \ (w + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots) \exp i \varepsilon^{-1} \phi$	defines $w_3 w_1, w_2$ etc
z_c	value of z for which $\omega = kU + lV$.
z_m	value of z for which $\omega - (kU + lV) = \pm N$
δz_i	thickness occupied by group at t_i
$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y}$	derivative following flow
$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$	derivative along ray
ε	small scaling parameter
$\bar{\rho}$	basic state density
ρ'	perturbation density
$\tau = \varepsilon t$	scaled time
$\sigma' = \frac{g\rho}{\bar{\rho}}$	buoyancy force
$\varepsilon^{-1} \phi$	eikonal
$\omega = \varepsilon^{-1} \frac{\partial \phi}{\partial t}$	angular frequency

Lecture Nine INTEGRAL STABILITY TECHNIQUES

1. Kelvin-Helmholtz instability

Consider a stratified shear flow in a non-rotating atmosphere. Then the linearised horizontal and vertical equations of motion for a two dimensional problem with basic state wind $U(z)$ and density $\rho_0(z)$ are

$$\rho_0(u_t + Uu_x + wU_z) = -p_x \tag{1}$$

$$\rho_0(w_t + Uw_x) = -p_z - \rho g \tag{2}$$

The continuity equation for incompressible flow becomes

$$u_x + w_z = 0 \tag{3}$$

and

$$\frac{D}{Dt}(\rho_0 + \rho) = 0, \text{ so that } \rho_t + U\rho_x + w\rho_{0z} = 0. \tag{4}$$

Introduce a stream function Ψ such that $u = \frac{\partial \Psi}{\partial z}$, $w = -\frac{\partial \Psi}{\partial x}$. Let ζ denote the deviation of an isopycnic from its basic state position. Then

$$\frac{D\zeta}{Dt} = \zeta_t + U\zeta_x = w = -\Psi_x \tag{5}$$

Assume perturbations of the form

$$\zeta = F(z) \exp ik(x-ct) \tag{6}$$

so that

$$\Psi = -(U-c)\zeta \tag{7}$$

$$u = -\frac{d}{dz}[(U-c)\zeta] \tag{8}$$

$$w = ik(U-c)\zeta \tag{9}$$

The horizontal equation of motion (1) may thus be written as

$$p = \rho_0(U-c)^2\zeta_z$$

which together with $\rho = -\zeta\rho_{0z}$ from (4) and substituting these into (2) gives

$$[\rho_0(U-c)^2\zeta_z]_z - g\zeta\rho_{0z} - k^2(U-c)^2 = 0$$

which may be written as

$$[\rho_0(U-c)^2 F_z]_z + \rho_0[\beta g - k^2(U-c)^2]F = 0 \tag{10}$$

where $\beta = -\frac{1}{\rho_0}\frac{d\rho_0}{dz}$ is a measure of the static stability of the atmosphere.

Yih (1965) discusses this equation in detail, but the conditions for instability may be derived from the theorem due to Miles (1960). The proof given here is due to Howard (1961) and illustrates the techniques of integral stability analysis.

Write

$$V = U - c, \quad G = V^{\frac{1}{2}}F \tag{11}$$

and assume that the disturbance is contained within the layer $0 \le z \le d$. Then $F(0) = F(d) = 0$ and so $G(0) = G(d) = 0$.

After some manipulation (10) becomes

$$(\rho_0 V G_z)_z - \left[\tfrac{1}{2}(\rho_0 U_z)_z + k^2\rho_0 V + \rho_0 V^{-1}(\tfrac{1}{4}U_z^2 - g\beta)\right]G = 0. \qquad (12)$$

Multiplying this by G^*, the complex conjugate of G, and integrating across the layer $0 \le z \le d$ gives

$$\int_0^d (\rho_0 V G_z)_z G^* - \left[\tfrac{1}{2}(\rho_0 U_z)_z + k^2\rho_0 V + \rho_0 V^{-1}(\tfrac{1}{4}U_z^2 - g\beta)\right]|G|^2\, dz = 0. \qquad (13)$$

Integrating the first term of this by parts gives

$$\int_0^d \rho_0 V G_z G_z^* + \left[\tfrac{1}{2}(\rho_0 U_z)_z + k^2\rho_0 V + \rho_0 V^{-1}(\tfrac{1}{4}U_z^2 - g\beta)\right]|G|^2 dz - \left[\rho_0 V G_z G^*\right]_0^d = 0 \qquad (14)$$

and applying the boundary conditions gives

$$\int_0^d \rho_0 V\left[|G_z|^2 + k^2|G|^2\right] + \tfrac{1}{2}(\rho_0 U_z)_z |G|^2 + \rho_0 V^*\left[\tfrac{1}{4}U_z^2 - g\beta\right]|G/V|^2\, dz = 0. \qquad (15)$$

Now, $c = c_r + ic_i$ and for the wave to be unstable $c_i \neq 0$, and since

$$Im(V) = -Im(V^*) = -c_i$$

taking the imaginary part of (15) gives

$$-c_i\left\{\int_0^d \rho_0\left[|G_z|^2 + k^2|G|^2\right]dz + \int_0^d \rho_0(\beta g - \tfrac{1}{4}U_z^2)\left|\frac{G}{V}\right|^2 dz\right\} = 0.$$

If c_i is to be non-zero, the sum of the integrals must be zero. The first integral is always non-negative, so that the integrand of the second must somewhere be less than zero for a growing wave to be supported. Thus

$$g\beta < \tfrac{1}{4}U_z^2 \qquad \text{somewhere in the flow.}$$

If the Richardson number is defined by

$$Ri = -g(d\rho/dz) \,/\, \rho_0 (dU/dz)^2$$

then the flow must be stable unless $Ri < \frac{1}{4}$ somewhere in the layer. Note that this is a necessary, but not sufficient, condition for instability.

Thorpe (1979) shows some laboratory simulations of Kelvin-Helmholtz waves.

2. Barotropic Instability

The barotropic vorticity equation was derived in lecture 1. This may be written in the form

$$\frac{D}{Dt}(\zeta + f) = 0 \qquad \text{, where } \zeta \text{ is the relative vorticity of the flow.}$$

Linearise this about the basic state $U(y)$. Then

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} - v\frac{\partial^2 U}{\partial y^2} + v\frac{\partial f}{\partial y} = 0.$$

Write $v = \psi_x$, $u = -\psi_y$, so that $\zeta = \nabla^2\psi$.

Substitute $\psi = \chi(y)\exp ik(x - ct)$, and write $\beta = \frac{\partial f}{\partial y}$. This yields

$$\chi_{yy} - \chi\left[k^2 - \frac{(\beta - U_{yy})}{(U - c)}\right] = 0 \qquad (1)$$

the Rayleigh stability equation

This may be analysed in the same way as before. Multiply by χ^* and integrate over the domain with assumed boundary conditions

$v = \psi_x = 0$ on $y = \pm a$. Then

$$\int_{-a}^{a} \chi^* \left[\chi_{yy} - \chi \left(k^2 - \frac{\beta - U_{yy}}{U-c} \right) \right] dy = 0$$

ie

$$\left[\chi_y \chi \right]_{-a}^{a} - \int_{-a}^{a} \chi_y^* \chi_y \, dy - \int_{-a}^{a} |\chi|^2 \left[k^2 - \frac{\beta - U_{yy}}{U-c} \right] dy = 0.$$

The first term is zero, so

$$\int_{-a}^{a} \left[|\chi_y|^2 + k^2 |\chi|^2 \right] dy + \int_{-a}^{a} |\chi|^2 \left(\frac{U_{yy} - \beta}{U - c_r - ic_i} \right) \left(\frac{U - c_r + ic_i}{U - c_r + ic_i} \right) dy = 0$$

which gives

$$\int_{-a}^{a} \left[|\chi_y|^2 + k^2 |\chi|^2 \right] dy + \int_{-a}^{a} |\chi|^2 \frac{(U_{yy} - \beta)}{|U-c|^2} (U - c_r + ic_i) \, dy = 0. \qquad (2)$$

Taking the imaginary part of this equation yields

$$c_i \int_{-a}^{a} |\chi|^2 \frac{(U_{yy} - \beta)}{|U-c|^2} dy = 0 \qquad (3)$$

so that for unstable waves the integral must vanish. This implies that $\beta - U_{yy}$ must change sign in $-a < y < a$. This requirement may be restated in terms of the profile of absolute vorticity $Z = \zeta - U_y$ of the basic state. For barotropic instability Z must have a maximum or a minimum value in the domain. Again this is a necessary but not sufficient condition for instability.

A further criterion may be derived from the real part of (2).

$$\int_{-a}^{a} \left(|\chi_y|^2 + k^2 |\chi|^2 \right) dy + \int_{-a}^{a} U |\chi|^2 \frac{(U_{yy} - \beta)}{|U-c|^2} dy = 0$$

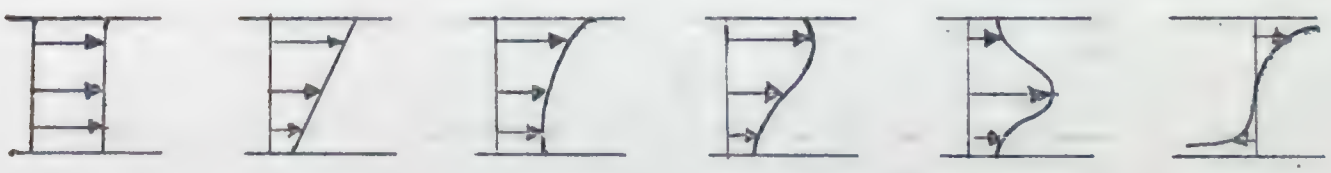
where (3) has been used to eliminate the G term. Define U_s to be the value of U where $\beta = U_{yy}$. Then

$$\int_{-a}^{a} \left[|\chi_y|^2 + k^2 |\chi|^2 \right] dy + \int_{-a}^{a} (U - U_s) |\chi|^2 \frac{(U_{yy} - \beta)}{|U-c|^2} dy = 0$$

since U_s is constant. Thus the condition for instability is that

$$(U - U_s)(\beta - U_{yy}) > 0$$

somewhere in $-a < y < a$, since the first integral is always positive. This is known as the Fjørtoft criterion.



Stable Stable Stable Unstable? Unstable? Stable

Stability of various wind distributions to barotropic instability.

Figure 1.

Thus a positive maximum of absolute vorticity in a westerly flow is a necessary condition for instability. Figure 1 shows some unstable profiles for the basic state.

It is possible to obtain more information from (1) by making substitutions of the form

$$\chi = F_n (U-c)^n$$

it can be shown (Dutton, 1976) that (1) becomes

$$(U-c)\left[-k^2 (U-c)^n F_n + \frac{\partial^2}{\partial y^2}\left[(U-c)^n F_n\right]\right] + (\beta - U_{yy})(U-c)^n F_n = 0. \tag{4}$$

Multiplying this by $(U-c)^{n-1} F_n^*$ and integrating over $-a<y<a$ leads to the equation

$$\int_{-a}^{a} (U-c)^{2n}\left[k^2 |F_n|^2 + \left|\frac{\partial F_n}{\partial y}\right|^2\right] - \beta (U-c)^{2n-1} |F_n|^2 + |F_n|^2 \left\{(U-c)^{2n-1} U_{yy}(1-n) + (U-c)^{2n-2} (U_y)^2 n(n-1)\right\} dy = 0. \tag{5}$$

Taking real and imaginary parts for $n=1$, and writing $Q_n = k^2 |F_n|^2 + \left|\frac{\partial F_n}{\partial y}\right|^2$ gives

$$\int_{-a}^{a} \left((U-c_r)^2 - c_i^2\right) Q_1 - \beta (U-c_r) |F_1|^2 dy = 0 \tag{6}$$

and

$$c_i \int_{-a}^{a} 2(U-c_r) Q_1 - \beta |F_1|^2 dy = 0 \tag{7}$$

If $c_i \neq 0$ then $c_r \leq U_{max}$, for otherwise the integral in (7) would be strictly negative. Dutton (ibid, exercise 14.3.8) states that for an arbitrary function g with suitable constraints applied

$$\int_{-a}^{a} \left|\frac{\partial g}{\partial y}\right|^2 dy \geq \frac{\pi^2}{4a^2} \int_{-a}^{a} |g|^2 dy.$$

Thus if β is constant (7) may be written as

$$\begin{aligned} \beta \int_{-a}^{a} |F_1|^2 dy &= \int_{-a}^{a} 2(U-c_r) Q_1 dy \\ &\geq \int_{-a}^{a} 2(U_{min} - c_r) Q_1 dy \\ &= 2(U_{min} - c_r) \int_{-a}^{a} Q_1 dy. \end{aligned}$$

Now $Q_n = k^2 |F_n|^2 + \left|\frac{\partial F_n}{\partial y}\right|^2$

so that, using the above

$$\int_{-a}^{a} Q_1 dy \geq \left[k^2 + \frac{\pi^2}{4a^2}\right] \int_{-a}^{a} |F_1|^2 dy.$$

Thus $\beta/2 \geq (U_{min} - c_r)\left(k^2 + \frac{\pi^2}{4a^2}\right)$.

Hence $$U_{min} - \beta/2 \left(k^2 + \frac{\pi^2}{4a^2}\right)^{-1} \leq c_r \leq U_{max} \tag{8}$$

Substituting (7) into (6) gives

$$\int_{-a}^{a} \left[U^2 - (c_r^2 + c_i^2)\right] Q_1 dy = \int_{-a}^{a} \beta U |F_1|^2 dy. \tag{9}$$

Note that

$$0 \geq \int_{-a}^{a} (U - U_{max})(U - U_{min}) Q_1 dy$$

so that

$$0 \geq \int_{-a}^{a} \left(\left(c_r - \frac{U_{max} + U_{min}}{2}\right)^2 + c_i^2\right) Q_1 + \beta \left(U - \frac{U_{max} + U_{min}}{2}\right) |F_1|^2 - \left(\frac{U_{max} - U_{min}}{2}\right)^2 Q_1 dy$$

after using (6), (7) and (9).

This may be simplified by noting that

$$U - \left(\frac{U_{max}+U_{min}}{2}\right) \geq -\left(\frac{U_{max}-U_{min}}{2}\right)$$

and so obtaining

$$\left(\frac{U_{max}-U_{min}}{2}\right)^2 + \frac{\beta}{2}\left(U_{max}-U_{min}\right)\frac{\int_{-a}^{a}|F_1|^2\,dy}{\int_{-a}^{a}Q_1\,dy} \geq \left(c_r - \frac{U_{max}+U_{min}}{2}\right)^2 + c_i^2$$

Finally $\int_{-a}^{a} Q_1\,dy \geq \left(k^2+\frac{\pi^2}{4a^2}\right)\int_{-a}^{a}|F_1|^2\,dy$

so that

$$\left(\frac{U_{max}-U_{min}}{2}\right)^2 + \frac{\beta}{2}\left(\frac{U_{max}-U_{min}}{k^2+\frac{\pi^2}{4a^2}}\right) \geq \left[c_r - \frac{U_{max}+U_{min}}{2}\right]^2 + c_i^2$$

This is known as the semi-circle theorem, and states that the complex phase speeds of unstable waves must lie within a semi-circle of radius R, where

$$R^2 = \left(\frac{U_{max}-U_{min}}{2}\right)^2 + \frac{\beta\left(U_{max}-U_{min}\right)}{2\left(k^2+\frac{\pi^2}{4a^2}\right)} \qquad (11)$$

and centred on $c_r = \frac{U_{max}+U_{min}}{2}$, $c_i = 0$ (12)

Note that since $c_r \leq U_{max}$ a portion of the semi-circle is not accessible to the solutions. Equation (10) gives an upper bound on the growth rate of unstable waves. If U does not vary, then (10) states that no unstable waves are possible.

Finally, set $\Lambda = \frac{1}{2}$ in (5) . Then

$$\int_{-a}^{a}(U-c)\,Q_{\frac{1}{2}} + \tfrac{1}{2}\left(U_{yy}-2\beta\right)|F_{\frac{1}{2}}|^2 + \tfrac{1}{4}\frac{(U_y)^2|F_{\frac{1}{2}}|^2}{|U-c|^2}\left(U-c^*\right)dy = 0.$$

Taking the imaginary part thus leads to

$$c_i\int_{-a}^{a} Q_{\frac{1}{2}} - \tfrac{1}{4}\frac{(U_y)^2}{|U-c|^2}|F_{\frac{1}{2}}|^2\,dy = 0$$

and for unstable waves the integral must vanish.

Using $|U-c|^2 \geq c_r^2$ gives

$$\left(k^2+\frac{\pi^2}{4a^2}\right)\int_{-a}^{a}|F_{\frac{1}{2}}|^2\,dy \leq \int_{-a}^{a} Q_{\frac{1}{2}}\,dy \leq \int_{-a}^{a}\tfrac{1}{4}(U_y)^2\,\frac{|F_{\frac{1}{2}}|^2}{c_i^2}\,dy$$

and hence

$$(kc_i)^2 \leq \frac{\max\left(U_y^2\right)}{4\left(1+\frac{\pi^2}{4a^2k^2}\right)} \qquad (13)$$

i.e. the maximum growth rate decreases as the horizontal length scale increases.

Summarising the above results for barotropic instability:

(i) barotropic instability can only occur if there is a positive maximum of absolute vorticity in a westerly flow,

(ii) unstable waves only occur in a flow with wind shear,

(iii) the complex phase speed of an unstable wave lies within a

semicircle given by (11) and (12), and the real phase speed is equal to the windspeed of the basic state somewhere ,

(iv) the growth rate is limited by the wind shear and the scale of motion (13).

Dutton (1976) derives the above results in the more general context of quasi-geostrophic theory. Drazin and Howard (1966) give a summary of this and other techniques.

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TABLE OF SYMBOLS USED IN LECTURE NINE

1.

$c = c_r + ic_i$	phase speed.
d	depth of layer containing disturbance.
$F(z)$	vertical structure function of perturbation.
g	gravitational acceleration.
$G(z)$	$V^{1/2} F$.
k	wave number.
p	pressure perturbation.
$Ri = \frac{g \frac{d\rho_o}{dz}}{\rho_o (\frac{dU}{dz})^2}$	Richardson number.
(u, w)	velocity perturbation.
$U(z)$	basic state velocity.
V	U-c.
$\beta = -\frac{1}{\rho_o} \frac{d\rho_o}{dz}$	stability parameter for flow
ρ	density perturbation.
$\rho_o(z)$	basic state density.
ψ	stream function.
ζ	displacement of isopycnic.
$( )^*$	complex conjugate.

2.

a	half-width of channel.
$c = c_r + ic_i$	phase speed.
f	Coriolis parameter.
$F_n(y) = (U-c)^n \chi(y)$	horizontal structure function.
$g(y)$	arbitrary function.
k	wave number.
$Q_n = k^2 \lvert F_n \rvert^2 + \lvert \frac{\partial F_n}{\partial y} \rvert^2$	
R	radius of semi-circle of instability.

(u, v)	velocity perturbation.
$U(y)$	basic state velocity.
U_s	value of U where $\beta = U_{yy}$.
$\bar{z} = f - U_y$	absolute vorticity.
$\beta = \frac{\partial f}{\partial y}$	variation of Coriolis parameter.
ψ	stream function.
ζ	relative vorticity.
$\chi(y)$	horizontal structure function.
$(\ )^*$	complex conjugate.

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